Universidade de Aveiro Departamento de Matemática
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Ana Paula Branco Nolasco

Propriedades de Regularidade de Operadores de Wiener-Hopf-Hankel

Regularity Properties of Wiener-Hopf-Hankel Operators

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tese apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica do Doutor Luís Filipe Pinheiro de Castro, Professor Associado com Agregação do Departamento de Matemática da Universidade de Aveiro

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Doutor Frank-Olme Ewal Speck
Professor Catedrático Convidado do Instituto Superior Técnico da Universidade Técnica de Lisboa

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palavras-chave

operadores de Wiener-Hopf-Hankel, propriedade de Fredholm, invertibilidade, factorização, função quase periódica, função semi-quase periódica, função quase periódica por troços

resumo

Nesta tese estudamos as propriedades de regularidade de operadores de Wiener-Hopf-Hankel com símbolos de Fourier pertencentes às álgebras das funções quase periódicas, das funções semi-quase periódicas e das funções quase periódicas por troços e consideramos estes operadores a actuar entre espaços de Lebesgue L^p (para $1 < p < \infty$). Por propriedades de regularidade entende-se invertibilidade lateral e bilateral, propriedade de Fredholm e solubilidade normal.

Propomos uma teoria de factorização para operadores de Wiener-Hopf-Hankel com símbolos de Fourier quase periódicos, e a actuar entre espaços de Lebesgue L^2, introduzindo uma factorização para os símbolos de Fourier quase periódicos de tal modo que as propriedades dos factores irão permitir correspondentes factorizações dos operadores. Um critério para a propriedade de semi-Fredholm e para a invertibilidade lateral e bilateral é assim obtido em termos de determinados índices das factorizações.

Baseado na relação delta após extensão, estabelecemos um teorema do tipo de Sarason para operadores de Wiener-Hopf-Hankel com símbolos de Fourier semi-quase periódicos, a actuar entre espaços de Lebesgue L^2. Uma generalização do teorema do tipo de Sarason é também obtida considerando agora os operadores a actuar entre espaços de Lebesgue L^p. Para operadores de Wiener-Hopf-Hankel com símbolos de Fourier quase periódicos por troços, a actuar entre espaços de Lebesgue L^2, um critério para a propriedade de Fredholm e para a invertibilidade lateral é também obtido através do uso da

relação delta após extensão. Todos estes resultados significam uma caracterização da propriedade de Fredholm e da invertibilidade lateral e bilateral destes operadores em termos dos valores médios e das médias geométricas dos representantes quase periódicos no infinito dos símbolos de Fourier, assim como das descontinuidades de determinadas funções auxiliares (no caso das funções quase periódicas por troços). Para cada caso, é apresentada uma fórmula para o índice de Fredholm.

Finalmente, de volta aos operadores de Wiener-Hopf-Hankel com símbolos de Fourier na subálgebra das funções quase periódicas APW, a actuar entre espaços de Lebesgue L^2, consideramos o caso mais geral de operadores de Wiener-Hopf-Hankel com símbolos matriciais de Fourier APW. Para estes operadores, obtemos um critério para a invertibilidade e a propriedade de semi-Fredholm baseado na hipótese de um específico conjunto de Hausdorff ser limitado fora de zero.

keywords

Wiener-Hopf-Hankel operators, Fredholm property, invertibility, factorization, almost periodic function, semi-almost periodic function, piecewise almost periodic function

abstract

In this thesis we study the regularity properties of Wiener-Hopf-Hankel operators with Fourier symbols belonging to the algebras of almost periodic, semi-almost periodic and piecewise almost periodic functions and we consider these operators acting between L^p Lebesgue spaces (for $1 < p < \infty$). By regularity properties one means one-sided and both-sided invertibility, Fredholm property and normal solvability.

We propose a factorization theory for Wiener-Hopf-Hankel operators with almost periodic Fourier symbols, and acting between L^2 Lebesgue spaces, by introducing a factorization concept for the almost periodic Fourier symbols such that the properties of the factors will allow corresponding operator factorizations. A criterion for the semi-Fredholm property and for one-sided and both-sided invertibility is therefore obtained upon certain indices of the factorizations.

Based on the delta relation after extension, we establish a Sarason's type theorem for Wiener-Hopf-Hankel operators with semi-almost periodic Fourier symbols and acting between L^2 Lebesgue spaces. We also derive a generalization of the Sarason's type theorem, the so-called Duduchava--Saginashvili's type theorem, when we consider the same kind of operators acting now between L^p Lebesgue spaces. For Wiener-Hopf-Hankel operators with piecewise almost periodic Fourier symbols, acting between L^2 Lebesgue spaces, a criterion for the Fredholm property and for the one-sided invertibility is also obtained upon the use of the delta relation after extension.

All these results mean a characterization of the Fredholm property, and one-sided invertibility of these operators, based on the mean motions and geometric mean values of the almost periodic representatives of the Fourier symbols at minus and plus infinity, as well as on the discontinuities of certain auxiliary functions (in the case of piecewise almost periodic functions). For each case, formulae for the Fredholm index of the operators are provided.

Finally, we return to Wiener-Hopf-Hankel operators with Fourier symbols in the subalgebra of almost periodic functions APW, acting between L^2 Lebesgue spaces, and we consider the more general case of Wiener-Hopf-Hankel operators with matrix APW Fourier symbols. For these operators we achieve an invertibility and semi-Fredholm criterion based on the assumption that a particular Hausdorff set is bounded away from zero.

Contents

List of Symbols

A^n	columns of length n with entries in A, 158
$A^{n\times n}$	$n \times n$ matrices with entries in A, 158
AP	class of almost periodic functions, 34
AP_p	closure of APW in $\mathcal{M}^p(\mathbb{R})$, 95
AP^-	smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ with $\lambda \le 0$, 41
AP^+	smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ with $\lambda \ge 0$, 41
APW	set of all almost periodic functions which can be written in the form of an absolutely convergent series, 40
APW^-	set of all functions $\psi \in APW$ such that $\Omega(\psi) \subset (-\infty, 0]$, 41
APW^+	set of all functions $\psi \in APW$ such that $\Omega(\psi) \subset [0, +\infty)$, 41
$\mathcal{A}_p\,(z_1, z_2)$	$= \left\{ z_1 + (z_2 - z_1)\,\sigma_p(\mu), \mu \in \overline{\mathbb{R}} \right\}$, 97
$\arg\phi$	continuous argument of ϕ, 39
$\mathbb{C}_+$	complex upper half plane $\Im m\, z > 0$, 15
$\mathbb{C}_-$	complex lower half plane $\Im m\, z < 0$, 15
$C(\dot{\mathbb{R}})$	set of all (bounded) continuous (complex-valued) functions on $\dot{\mathbb{R}}$, 41
$C(\overline{\mathbb{R}})$	set of all (bounded) continuous (complex-valued) functions on $\mathbb{R}$ with a possible jump at ∞, 42
$C_0(\dot{\mathbb{R}})$	set of all functions in $C(\dot{\mathbb{R}})$ such that the limits at $-\infty$ and at $+\infty$ are equal to zero, 42
$C_p(\dot{\mathbb{R}})$	closure of $C(\dot{\mathbb{R}})$ in $\mathcal{M}^p(\mathbb{R})$, 95
$C_p(\overline{\mathbb{R}})$	closure of $C(\overline{\mathbb{R}})$ in $\mathcal{M}^p(\mathbb{R})$, 95
$\operatorname{Coker} T$	cokernel of the operator T, 8
$\mathbb{D}_+$	$= \{z \in \mathbb{C} : \lvert z\rvert < 1\}$, 24
$\mathbb{D}_-$	$= \{z \in \mathbb{C} : \lvert z\rvert \ge 1\} \cup \{\infty\}$, 24
$d(T)$	dimension of the cokernel of the operator T, 8
$\mathbf{d}(\phi)$	geometric mean value of $\phi \in \mathcal{G}AP$, 40
$\mathbf{d}_l(\phi)$	left geometric mean value of $\phi \in \mathcal{G}PAP$, 47
$\mathbf{d}_r(\phi)$	right geometric mean value of $\phi \in \mathcal{G}PAP$, 47

e_λ	$e_\lambda(x) = e^{i\lambda x}$ $(x \in \mathbb{R})$, 36
$\operatorname{ess\,sup}$	essential supremum, 2
$\mathcal{F}$	Fourier transformation, 3
$\mathcal{F}^{-1}$	inverse of the Fourier transformation $\mathcal{F}$, 3
ϕ_l	almost periodic representative of ϕ at $-\infty$, 45, 47
ϕ_r	almost periodic representative of ϕ at $+\infty$, 45, 47
$\mathcal{G}B$	group of all invertible elements of a Banach algebra B, 17
H_ϕ	Hankel operator, 7
$\mathcal{H}(\Theta)$	Hausdorff set or numerical range of the complex matrix $\Theta \in \mathbb{C}^{n\times n}$, 159
$H^\infty(\mathbb{C}_\pm)$	set of all bounded and analytic functions in $\mathbb{C}_\pm$, 15
$H^\infty(\mathbb{D}_\pm)$	set of all bounded and analytic functions in $\mathbb{D}_\pm$, 24
$H^p(\mathbb{C}_\pm)$	set of all functions ϕ which are analytic in $\mathbb{C}_\pm$ and such that $\sup_{\pm y>0} \int_{\mathbb{R}} \lvert\phi(x+iy)\rvert^p\, dy < \infty$, 15
$H^p(\mathbb{D}_+)$	set of all functions ϕ which are analytic in $\mathbb{D}_+$ and such that $\sup_{r\in(0,1)} \int_0^{2\pi} \lvert\phi(re^{i\theta})\rvert^p\, d\theta < \infty$, 24
$H^p(\mathbb{D}_-)$	set of all functions $\phi(z)$ $(z \in \mathbb{D}_-)$ for which $\phi(\frac{1}{z})$ is a function in $H^p(\mathbb{D}_+)$, 24
$H^\infty_\pm(\mathbb{R})$	set of all functions in $L^\infty(\mathbb{R})$ that are non-tangential limits of elements in $H^\infty(\mathbb{C}_\pm)$, 15
$H^p_\pm(\mathbb{R})$	set of all functions in $L^p(\mathbb{R})$ that are non-tangential limits of elements in $H^p(\mathbb{C}_\pm)$, 15
$H^p_\pm(\mathbb{T})$	set of all functions in $L^p(\mathbb{T})$ that are non-tangential limits of elements in $H^p(\mathbb{D}_\pm)$, 24
$\operatorname{Im} T$	image of the operator T, 8
$\Im m(z)$	imaginary part of the complex number z
$\operatorname{Ind} T$	Fredholm index of the operator T, 9
$\operatorname{ind} \phi$	Cauchy index of ϕ, 103, 105, 108, 135
$\inf$	infimum
I_Z	identity operator on the Banach space Z, 7
J	reflection operator on $\mathbb{R}$, 5, 158
$J_{\mathbb{T}}$	reflection operator on $\mathbb{T}$, 25
$\operatorname{Ker} T$	kernel of the operator T, 8
$\kappa(\phi)$	mean motion of $\phi \in \mathcal{G}AP$, 39
$\kappa_l(\phi)$	left mean motion of $\phi \in \mathcal{G}PAP$, 47
$\kappa_r(\phi)$	right mean motion of $\phi \in \mathcal{G}PAP$, 47
ℓ_0	zero extension operator from $L^p(\mathbb{R}_+)$ into $L^p(\mathbb{R})$, 2
ℓ^2	space of all infinite sequences $\{\xi_i\}_{i=1}^\infty$ such that $\sum\limits_{i=1}^{\infty} \lvert\xi_i\rvert^2 < \infty$, 6
ℓ^e	even extension operator from $L^p(\mathbb{R}_+)$ into $L^p(\mathbb{R})$, 7
ℓ^o	odd extension operator from $L^p(\mathbb{R}_+)$ into $L^p(\mathbb{R})$, 7
$\mathcal{L}(X,Y)$	linear space of all bounded linear operators from the Banach space X into the Banach space Y
$L^\infty(\Omega)$	Lebesgue space of all measurable and essentially bounded functions on Ω, 2

$L^p(\Omega)$	Lebesgue space of p^{th}-power-integrable functions on Ω, 2
$L^p_+(\mathbb{R})$	subspace of $L^p(\mathbb{R})$ formed by all the functions supported in the closure of $\mathbb{R}_+$, 2
$L^p_-(\mathbb{R})$	subspace of $L^p(\mathbb{R})$ formed by all the functions supported in the closure of $\mathbb{R}_-$, 2
$L^p_{\text{even}}(\mathbb{T})$	set of all functions $\phi \in L^p(\mathbb{T})$ such that $\phi(t) = \phi(t^{-1})$, 58
$L^p_{J_\mathbb{T}}(\mathbb{T})$	$= \operatorname{Im} P_{J_\mathbb{T}}\|_{L^p(\mathbb{T})}$, 58
$M(\phi)$	Bohr mean value of ϕ (or mean value of ϕ), 36
$\mathcal{M}^p(\mathbb{R})$	set of all Fourier multipliers on $L^p(\mathbb{R})$, 6
$n(T)$	dimension of the kernel of the operator T, 8
$\mathbb{N}_0$	$= \{0, 1, 2, 3, \ldots\}$, 144
Ω	open set of $\mathbb{R}$, 2
$\Omega(\phi)$	Bohr-Fourier spectrum of ϕ, 38
$\mathcal{P}$	set of all trigonometric polynomials on $\mathbb{T}$, 30
P	orthogonal projection of $L^2(\mathbb{R})$ onto $H^2_+(\mathbb{R})$, 24
$P_\mathbb{T}$	orthogonal projection of $L^2(\mathbb{T})$ onto $H^2_+(\mathbb{T})$, 24
P_+	canonical projector from $L^p(\mathbb{R})$ onto $L^p_+(\mathbb{R})$, 2
P_-	canonical projector from $L^p(\mathbb{R})$ onto $L^p_-(\mathbb{R})$, 2
$P_{J_\mathbb{T}}$	$= \frac{I+J_\mathbb{T}}{2}$, 58
PAP	algebra of piecewise almost periodic functions on $\mathbb{R}$, 47
PC	algebra of piecewise continuous functions on $\dot{\mathbb{R}}$, 46
PC_0	class of all piecewise continuous functions for which both limits at $-\infty$ and at $+\infty$ are equal to zero, 46
$PC(\mathbb{T})$	algebra of piecewise continuous functions on $\mathbb{T}$, 85
$\psi^\#$	$\psi^\# : \dot{\mathbb{R}} \times [0,1] \to \mathbb{C}$, $\psi^\#(x,\mu) = (1-\mu)\,\psi(x-0) + \mu\,\psi(x+0)$, 96
ψ^p	$\psi^p : \dot{\mathbb{R}} \times \overline{\mathbb{R}} \to \mathbb{C}$, $\psi^p(x,\mu) = (1-\sigma_p(\mu))\,\psi(x-0) + \sigma_p(\mu)\,\psi(x+0)$, 98
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	positive half-line, 2
$\mathbb{R}_-$	negative half-line, 2
$\dot{\mathbb{R}}$	$= \mathbb{R} \cup \{\infty\}$, 41
$\overline{\mathbb{R}}$	$= \mathbb{R} \cup \{\pm\infty\}$, 42
$\mathcal{R}(\phi)$	essential range of ϕ, 28
$\Re e(z)$	real part of the complex number z
r_+	restriction operator from $L^p(\mathbb{R})$ into $L^p(\mathbb{R}_+)$, 2
S	Cauchy singular integral operator on $L^2(\mathbb{R})$, 24
$S_\mathbb{T}$	Cauchy singular integral operator on $L^2(\mathbb{T})$, 24
$\mathcal{S}(\mathbb{R})$	Schwartz space of all rapidly decreasing functions on $\mathbb{R}$, 3
SAP	algebra of the semi-almost periodic functions, 42
SAP_p	smallest closed subalgebra of $\mathcal{M}^p(\mathbb{R})$ that contains AP_p and $C_p(\overline{\mathbb{R}})$, 96
$sp\,T$	spectrum of the operator $T \in \mathcal{L}(X)$, 28
$sp_{ess}T$	essential spectrum of the operator $T \in \mathcal{L}(X)$, 28
σ_p	$\sigma_p(\mu) = \frac{1}{2} + \frac{1}{2}\coth\left[\pi\left(\frac{i}{p} + \mu\right)\right]$ $(\mu \in \overline{\mathbb{R}})$, 97

$sp_X\,\phi$	spectrum of the element $\phi \in X$, 27
sup	supremum
$\mathbb{T}$	unit circle, 3
$\mathbb{T}_+$	$= \{t \in \mathbb{T} : \Im m\, t > 0\}$, 88
$(T{+}H)_\nu$	Toeplitz plus Hankel operator, 25
$(T{-}H)_\nu$	Toeplitz minus Hankel operator, 25
$\mathbb{W}$	Wiener algebra, 4
W_ϕ	Wiener-Hopf operator, 7
$(W{+}H)_\phi$	Wiener-Hopf plus Hankel operator, 7
$(W{-}H)_\phi$	Wiener-Hopf minus Hankel operator, 7
$\mathrm{wind}\,\phi$	winding number of ϕ, 104, 105, 108, 135, 137, 138
$\lVert.\rVert_A$	norm in the linear space A
$f * g$	convolution of f and g, 3
$\widetilde{\varphi}$	action of the reflection operator on the function φ, 5
$\sim$	equivalence relation, 11
$\overset{*}{\sim}$	equivalence after extension relation, 11
$\oplus$	direct sum
∂A	boundary of the set A
$[c_1, c_2]$	line segment in the complex plane between and including the endpoints $c_1, c_2 \in \mathbb{C}$, 78
$\{x\}$	fractional part $\mu \in [0, 1)$ of the real number $x = n + \mu$, with $n \in \mathbb{Z}$, 105

Introduction

Wiener-Hopf-Hankel operators (as well as their discrete analogues based on Toeplitz and Hankel operators) are well-known to play an important role in several applied areas. E.g., this is the case in certain wave diffraction problems, digital signal processing, discrete inverse scattering, and linear prediction. For concrete examples of a detailed enrollment of those operators in these (and others) applications we refer to [21, 23, 32, 49, 52, 53, 68, 74].

In view of the needs of the applications, it is natural to expect a corresponding higher interest in mathematical fundamental research for those kind of operators. In fact, in recent years several authors have contributed to the mathematical understanding of Wiener-Hopf--Hankel operators (and their discrete analogues) under different types of assumptions (cf. [6, 7, 8, 20, 22, 31, 41, 42, 52, 63, 64, 69]). As a consequence, the theory of Wiener-Hopf-Hankel operators is nowadays well developed for some classes of Fourier symbols (like in the case of continuous or piecewise continuous symbols). In particular, the invertibility and Fredholm properties of such kind of operators with piecewise continuous Fourier symbols are now well known (and of great importance for the applications [23, 49, 52, 74]). However, this is not the case for almost periodic or semi-almost periodic Fourier symbols which are also important in the applications in view of their appearance due to, e.g., (i) particular finite boundaries in the geometry of physical problems [21], or (ii) the needs of compositions with shift operators which introduce almost periodic and semi-almost periodic elements in the Fourier symbols of those operators [19, 44].

Having this in mind, in this thesis we study the regularity properties of Wiener-Hopf--Hankel operators such that the Wiener-Hopf operator, and the Hankel operator have equal or symmetric Fourier symbols, and we consider these operators having Fourier symbols belonging to the algebras of almost periodic, semi-almost periodic and piecewise almost periodic functions and acting between L^p Lebesgue spaces ($1 < p < \infty$). In the case where both Wiener-Hopf and Hankel operators have the same Fourier symbol, we obtain a Wiener-Hopf plus Hankel operator, and in the case where Wiener-Hopf and Hankel operators have symmetric Fourier symbols, we get a Wiener-Hopf minus Hankel operator.

To study the regularity properties of Wiener-Hopf-Hankel operators, we use certain relations between operators that allow us derive the regularity properties of the Wiener--Hopf-Hankel operators from the regularity properties of others operators for which results concerning to regularity properties are known.

This thesis is organized as follows. In the first chapter, after giving the formal definition of the operators under study - the Wiener-Hopf-Hankel operators - several relations between some classes of convolution type operators are investigated. The first relation is a multiplicative relation between Wiener-Hopf plus Hankel operators, which is a generalization of the well-known result for the Wiener-Hopf operators. Using the same reasoning, an analogue multiplicative relation between Wiener-Hopf minus Hankel operators is also possible to derive. After this, we present a relation between Wiener-Hopf plus Hankel operators and Wiener-Hopf operators (based on Wiener-Hopf minus Hankel and paired operators). Since this relation is in fact a Δ–relation after extension between the Wiener--Hopf plus Hankel operator and the Wiener-Hopf operator, and therefore there is transfer of the regularity properties from the Wiener-Hopf operator to the Wiener-Hopf plus Hankel operator, this relation turns out to be a decisive result in the study of the regularity properties of the Wiener-Hopf-Hankel operators. The last relations under investigation in this chapter are two relations of equivalence between Wiener-Hopf plus/minus Hankel operators and Toeplitz plus/minus Hankel operators, this due to the equivalence relation existent between Wiener-Hopf and Toeplitz operators. Finally, we use these equivalence relations between Wiener-Hopf-Hankel and Toeplitz-Hankel operators to derive necessary conditions for the semi-Fredholm property of Wiener-Hopf-Hankel operators.

Chapter 2 is devoted to the several algebras where the Fourier symbols of the Wiener-

-Hopf-Hankel operators in study belong to. Here it will be presented the definition of the algebra of almost periodic functions, the algebra of semi-almost periodic functions, and the algebra of piecewise almost periodic functions. Some properties of these algebras are presented as well.

The main purpose of Chapter 3 is to establish an invertibility and semi-Fredholm criterion for Wiener-Hopf plus Hankel operators with almost periodic Fourier symbols, and acting between L^2 Lebesgue spaces. We start by considering the case of Wiener-Hopf plus Hankel operators with Fourier symbols in the subalgebra of almost periodic functions APW and then we consider the more general case of Wiener-Hopf plus Hankel operators with almost periodic Fourier symbols. To obtain the invertibility and semi-Fredholm criterion we introduce a factorization concept for the almost periodic Fourier symbols such that the properties of the factors will allow corresponding operator factorizations. Using then the multiplicative relation presented in Chapter 1, a criterion for the semi-Fredholm property and for the one-sided and both-sided invertibility is therefore obtained upon certain indices of the factorizations. Under such conditions, the one-sided and two-sided inverses of the operators are also obtained. Moreover, the introduced factorizations also allow the exposition of dependencies between the invertibility of Wiener-Hopf and Wiener-Hopf plus Hankel operators with the same Fourier symbol. All these results hold true for Wiener-Hopf minus Hankel operators since we use an analogue multiplicative relation for Wiener-Hopf minus Hankel operators of that one used for Wiener-Hopf plus Hankel operators.

In Chapter 4, Wiener-Hopf-Hankel operators with semi-almost periodic Fourier symbols are in focus. Considering in first place these operators acting between L^2 Lebesgue spaces, and motivated by the Sarason's Theorem, we obtain a generalization of the invertibility and semi-Fredholm criteria achieved in Chapter 3 for Wiener-Hopf-Hankel operators with almost periodic symbols - the so-called Sarason's type theorem. This result is attained by using the Δ–relation after extension and also the main idea of the approach of R. V. Duduchava and A. I. Saginashvili [29]. After settled the Sarason's type theorem, a reformulation of the invertibility and semi-Fredholm criterion for Wiener-Hopf plus Hankel operators with almost periodic symbols in terms of the value of the mean motion of the Fourier symbol of the operator is presented. Moreover, we generalize this last result by providing an invertibility and semi-Fredholm criterion for Wiener-Hopf-Hankel operators with almost periodic Fourier symbols without the assumption on the corresponding factorization of the Fourier symbol of the operator, being this also possible due to the Sarason's type theorem. In a second part of this chapter, we consider Wiener-Hopf-Hankel operators with semi-almost periodic Fourier symbols acting between L^p Lebesgue spaces $(1 < p < \infty)$. Having by motivation the Duduchava-Saginashvili's theorem, we achieve a generalization of the Sarason's type theorem, the so-called Duduchava-Saginashvili's type theorem. The fundamental key to reach this result is the Δ–relation after extension between the Wiener-Hopf plus Hankel operator and the Wiener-Hopf operator. All the results mentioned above mean a characterization of the Fredholm property, and one-sided invertibility of the Wiener-Hopf-Hankel operators, based on the mean motions and geometric mean values of the almost periodic representatives of the Fourier symbols at minus and plus infinity. We end up this chapter with a section devoted to the Fredholm index of the Wiener-Hopf-Hankel operators under study in this chapter. In the first case where the operators are considered acting between L^2 Lebesgue spaces, a Fredholm index formula is derived in terms of the winding number of the Fourier symbols of the operators, while in the second case where we consider the operators acting between L^p Lebesgue spaces, the Fredholm index formula is given in terms of the winding number of continuous functions constructed from the Fourier symbols of the operators. Additionally, in the case of Wiener--Hopf-Hankel operators acting between L^2 Lebesgue spaces, conditions for the invertibility of these operators are also obtained.

Chapter 5 presents a criterion for the semi-Fredholm property and for the one-sided invertibility of Wiener-Hopf-Hankel operators with piecewise almost periodic Fourier symbols, acting between L^2 Lebesgue spaces. This criterion is also obtained upon the use of the Δ–relation after extension and, in virtue of the nature of piecewise almost periodic functions, it is based on the mean motions and geometric mean values of the almost periodic representatives of the Fourier symbols as well as on the discontinuities of certain auxiliary functions. After established the one-sided invertibility and semi-Fredholm criteria, we obtain a formula for the sum of the Fredholm indices of the Wiener-Hopf plus and minus Hankel operators. Since in this chapter we are dealing with operators having discontinuous symbols, the Fredholm index formula is also interpreted upon different cases of symmetries of the discontinuities of the Fourier symbols. In order to exemplify the simplification that occurs in the Fredholm index formula due to the symmetries of the discontinuities of the Fourier symbol, several examples are presented. Finally, the (both-sided) invertibility of the operators in study is also discussed.

In the last chapter, motivated by a result due to R. G. Babadzhanyan and V. S. Rabinovich that settles the (one-sided and both-sided) invertibility and the semi-Fredholm property of Wiener-Hopf operators with matrix APW Fourier symbols having Hausdorff sets bounded away from zero, we derive a generalization of the invertibility and semi--Fredholm criteria obtained in Chapter 3. Here, we consider Wiener-Hopf-Hankel operators acting in L^2 Lebesgue spaces with matrix APW Fourier symbols having a particular Hausdorff set bounded away from zero and obtain for these operators an invertibility and semi-Fredholm criteria. Similarly to the result obtained in Theorem 3.2.1, the invertibility and semi-Fredholm criteria here presented is given in terms of the mean motion, in this case, depending on a particular Hausdorff set bounded away from zero (instead of the mean motion of the Fourier symbol of the operator as in Theorem 3.2.1).

As far as the author knows, the results presented in this thesis are new. Most of the material is published or accepted for publication in journals or conference proceedings [58, 59, 60, 61], and the material which is not published or accepted for publication is submitted for publication in journals [17, 57]. Auxiliary results of other authors included in the text are properly referred to.

Chapter 1

Convolution Type Operators

The study of relations between different classes of operators is an important subject in Operator Theory since, in particular, it allows the transfer of properties from one class of operators to the other ones. Due to this transfer of properties and in order to achieve the invertibility and semi-Fredholm criteria presented in Chapters 3, 4, 5 and 6, the main purpose of this chapter is the study of some relations between some classes of convolution type operators. In first place, we present a multiplicative relation between Wiener-Hopf plus Hankel operators (which in fact is a corresponding result for Wiener-Hopf plus Hankel operators of the well-known result for the Wiener-Hopf operators). In second place, it is exhibited a relation between Wiener-Hopf plus Hankel operators and Wiener-Hopf operators (based on Wiener-Hopf minus Hankel and paired operators). This relation will perform a significant role in the obtainment of several results because it allows to study the regularity properties of Wiener-Hopf operators, and then transfer the regularity properties to Wiener-Hopf-Hankel operators. Finally, in what concerns to relations between convolution type operators, since Wiener-Hopf and Toeplitz operators in Hilbert spaces are equivalent operators, it is also presented a way to relate Wiener-Hopf-Hankel operators with Toeplitz-Hankel operators. Here and in what follows, we will simply call Wiener--Hopf-Hankel operators to both Wiener-Hopf plus Hankel, and Wiener-Hopf minus Hankel operators (cf. also [49, 52, 74]). Similarly, by Toeplitz-Hankel operators we mean both

Toeplitz plus Hankel, and Toeplitz minus Hankel operators. Using equivalence relations between Wiener-Hopf-Hankel and Toeplitz-Hankel operators, we end up this chapter by deriving necessary conditions for the semi-Fredholm property of Wiener-Hopf-Hankel operators.

Before presenting the results mentioned above, we will first introduce the formal definitions of the operators under study as well as some of the notation that will be used throughout this thesis.

1.1 Some definitions, notations, and historical notes

Let Ω be an open set of $\mathbb{R}$. For $1 \leq p < \infty$, $L^p(\Omega)$ represents the Banach space of Lebesgue measurable complex-valued functions φ on Ω such that $|\varphi|^p$ is integrable. This space is endowed with the norm

$$\|\varphi\|_{L^p(\Omega)} := \left(\int_\Omega |\varphi(\eta)|^p d\eta \right)^{\frac{1}{p}}. \tag{1.1.1}$$

Furthermore, let $L^p_+(\mathbb{R})$ denote the subspace of $L^p(\mathbb{R})$ formed by all the functions supported in the closure of $\mathbb{R}_+ := (0, +\infty)$, and $L^p_-(\mathbb{R})$ represent the subspace of $L^p(\mathbb{R})$ formed by all the functions supported in the closure of $\mathbb{R}_- := (-\infty, 0)$. We will use the *canonical projectors* P_+ and P_- that map $L^p(\mathbb{R})$ onto $L^p_+(\mathbb{R})$, and $L^p(\mathbb{R})$ onto $L^p_-(\mathbb{R})$, respectively. Considering r_+ being the *restriction operator* from $L^p(\mathbb{R})$ into $L^p(\mathbb{R}_+)$ and ℓ_0 the *zero extension operator* from $L^p(\mathbb{R}_+)$ into $L^p(\mathbb{R})$, we have

$$P_+ = \ell_0 r_+. \tag{1.1.2}$$

Consider also $L^\infty(\Omega)$ as the Banach space of Lebesgue measurable and essentially bounded (complex-valued) functions on Ω. The norm in $L^\infty(\Omega)$ is given by

$$\|\phi\|_{L^\infty(\Omega)} := \operatorname{ess\,sup} |\phi(\eta)|, \tag{1.1.3}$$

where

$$\operatorname{ess\,sup} |\phi(\eta)| := \inf\{\alpha : |\{\eta \in \Omega : |\phi(\eta)| > \alpha\}| = 0\}. \tag{1.1.4}$$

For the case of Lebesgue spaces defined on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, let $L^p(\mathbb{T})$ $(1 \leq p < \infty)$ stand for the Banach space of Lebesgue measurable complex-valued functions φ on $\mathbb{T}$ such that $|\varphi|^p$ is integrable. The norm in $L^p(\mathbb{T})$ is given by

$$\|\varphi\|_{L^p(\mathbb{T})} := \frac{1}{2\pi}\left(\int_0^{2\pi} |\varphi(e^{i\theta})|^p d\theta\right)^{\frac{1}{p}}. \tag{1.1.5}$$

Moreover, let $L^\infty(\mathbb{T})$ denote the Banach space of Lebesgue measurable and essentially bounded (complex-valued) functions defined on $\mathbb{T}$. $L^\infty(\mathbb{T})$ is endowed with the essential supremum norm

$$\|\phi\|_{L^\infty(\mathbb{T})} := \operatorname{ess\,sup}|\phi(\eta)|, \tag{1.1.6}$$

where, in this case,

$$\operatorname{ess\,sup}|\phi(\eta)| := \inf\{\alpha : |\{\eta \in \mathbb{T} : |\phi(\eta)| > \alpha\}| = 0\}. \tag{1.1.7}$$

Let $\mathcal{F}$ denote the *Fourier transformation* defined in the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions on $\mathbb{R}$ by

$$\mathcal{F}\varphi(\xi) := \int_{\mathbb{R}} e^{i\xi\eta}\varphi(\eta)\, d\eta, \qquad \xi \in \mathbb{R}\,, \tag{1.1.8}$$

and let $\mathcal{F}^{-1}$ denote the *inverse of the Fourier transformation* $\mathcal{F}$, also defined on $\mathcal{S}(\mathbb{R})$, given by

$$\mathcal{F}^{-1}\psi(\eta) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi\eta}\psi(\xi)\, d\xi, \quad \eta \in \mathbb{R}\,. \tag{1.1.9}$$

A *convolution operator* is defined by

$$\mathcal{C}\varphi(\xi) := (\mathcal{K} * \varphi)(\xi) = \int_{\mathbb{R}} \mathcal{K}(\xi - \eta)\varphi(\eta)\, d\eta, \qquad \xi \in \mathbb{R}, \tag{1.1.10}$$

where $\mathcal{K}$ is called the *convolution kernel* of $\mathcal{C}$. Convolution operators may be considered acting in several functions spaces as well as their convolution kernels that may also be considered in different functions spaces. For instance, we may consider the convolution operator $\mathcal{C}$ acting in $L^1(\mathbb{R})$ with $\mathcal{K} \in L^1(\mathbb{R})$. The convolution operator $\mathcal{C}$ can be written in the form

$$\mathcal{C} = \mathcal{F}^{-1}\phi_{\mathcal{C}} \cdot \mathcal{F}, \tag{1.1.11}$$

with $\phi_{\mathcal{C}} = \mathcal{F}\mathcal{K}$. From this, we see why $\phi_{\mathcal{C}}$ is called the *Fourier symbol* of the convolution operator $\mathcal{C}$. If in (1.1.10) instead of $\mathbb{R}$ we consider the open set $\Omega \subset \mathbb{R}$, then we obtain a *convolution type operator*

$$\mathcal{A}\varphi(\xi) := (\mathcal{K} * \varphi)(\xi) = \int_{\Omega} \mathcal{K}(\xi - \eta)\varphi(\eta)\, d\eta, \qquad \xi \in \Omega. \tag{1.1.12}$$

The probably best known convolution type operators are the Wiener-Hopf operators acting between Lebesgue spaces on the half-line. We recall that the name "Wiener-Hopf operators" is due to the initial work of N. Wiener and E. Hopf in 1931 [76] where a reasoning to solve integral equations whose kernels depend only on the difference of the arguments was provided:

$$cf(x) + \int_0^{+\infty} k(x-y)f(y)dy = g(x), \quad x \in \mathbb{R}_+\,, \tag{1.1.13}$$

i.e. the so-called *integral Wiener-Hopf equations*. Here $c \in \mathbb{C}$, $k \in L^1(\mathbb{R})$ and $f, g \in L^2(\mathbb{R}_+)$, where c and k are fixed, g is given and f is the unknown element.

From those Wiener-Hopf equations arise the *(classical) Wiener-Hopf operators* defined by

$$W_\phi f(x) = cf(x) + \int_0^{+\infty} k(x-y)f(y)dy \ , \quad x \in \mathbb{R}_+\,, \tag{1.1.14}$$

where $\phi = c + \mathcal{F}k$ belongs to the Wiener algebra. The *Wiener algebra* is defined by

$$\mathbb{W} := \{\phi : \phi = c + \mathcal{F}k, c \in \mathbb{C}, k \in L^1(\mathbb{R})\} \tag{1.1.15}$$

and it is a Banach algebra when endowed with the norm

$$\|c + \mathcal{F}k\|_{\mathbb{W}} := |c| + \|k\|_{L^1(\mathbb{R})} \tag{1.1.16}$$

and the usual multiplication operation. The Wiener algebra is a subalgebra of $L^\infty(\mathbb{R})$. Having in mind the convolution operation, the definition of the classical Wiener-Hopf operators gives rise to the following representation of the Wiener-Hopf operators

$$W_\phi := r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F} \ : \ L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+)\,. \tag{1.1.17}$$

Making use of the canonical projector on $L^2_+(\mathbb{R})$, the Wiener-Hopf operators on $L^2_+(\mathbb{R})$ may also be written in the form

$$P_+A = \ell_0 r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F} : L^2_+(\mathbb{R}) \to L^2_+(\mathbb{R})\,, \tag{1.1.18}$$

where A is the translation invariant operator $\mathcal{F}^{-1}\phi \cdot \mathcal{F}$. Looking now to the structure of the operators in (1.1.17) and (1.1.18), we recognize that possibilities other than only the Wiener algebra can be considered for the so-called Fourier symbols ϕ of the Wiener-Hopf operators. Namely, we may consider to choose ϕ among the $L^\infty(\mathbb{R})$ elements.

Within the context of (1.1.13) and (1.1.14), the *Hankel integral operators* have the form

$$Hf(x) = \int_0^{+\infty} k(x+y)f(y)dy \ , \quad x \in \mathbb{R}_+ \tag{1.1.19}$$

for some $k \in L^1(\mathbb{R})$. In this case, it is well-known that H, as an operator defined between L^2 spaces, is a compact operator. However, as seen above, it is also possible to provide a rigorous meaning to the expression (1.1.19) when the kernel k is a temperate distribution whose Fourier transform belongs to $L^\infty(\mathbb{R})$. In such case, Hankel operators admit the representation

$$H_\phi := r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}J \ : \ L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+) \, . \tag{1.1.20}$$

Here and in what follows, J is the *reflection operator on* $\mathbb{R}$ given by the rule

$$J\varphi(x) = \widetilde{\varphi}(x) = \varphi(-x), \ x \in \mathbb{R} \, . \tag{1.1.21}$$

Like in (1.1.18), we may write Hankel operators on $L^2_+(\mathbb{R})$ as

$$P_+AJ = \ell_0 r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}J : L^2_+(\mathbb{R}) \to L^2_+(\mathbb{R}) \, , \tag{1.1.22}$$

being $A = \mathcal{F}^{-1}\phi \cdot \mathcal{F}$. We would like to mention that the discrete analogue of H has its roots in the year of 1861 with the Ph.D. thesis of H. Hankel [39]. There the study of finite matrices with entries depending only on the sum of the coordinates was proposed. Determinants of infinite complex matrices with entries defined by $a_{jk} = a_{j+k}$ (for $j,k \geq 0$, and where $a = \{a_j\}_{j\geq 0}$ is a sequence of complex numbers) were also studied. For these (infinite) Hankel matrices, one of the first main results was obtained by L. Kronecker in 1881 [47] when characterizing the Hankel matrices of finite rank as the ones that have corresponding power series, $a(z) = \sum_{j=0}^{\infty} a_j \, z^j$, which are rational functions. In 1906, D. Hilbert proved that the operator (induced by the famous Hilbert matrix),

$$\mathcal{H} : \ell^2 \to \ell^2 \ , \ \{b_j\}_{j\geq 0} \mapsto \left\{ \sum_{k=0}^{\infty} \frac{b_k}{j+k+1} \right\}_{j \geq 0} \ , \tag{1.1.23}$$

is bounded on the space ℓ^2 of all infinite sequences $\{\xi_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} |\xi_i|^2 < \infty$. This result may be viewed as the origin of (discrete) Hankel operators, as natural objects arising from Hankel matrices. Later on, in 1957, Z. Nehari presented a characterization of bounded Hankel operators on ℓ^2 [55]. Due to the importance of such characterization, we may say that it marks the beginning of the contemporary period of the study of Hankel operators.

After a brief note on the origins of Wiener-Hopf and Hankel operators and before present the definition of Wiener-Hopf plus Hankel operators and Wiener-Hopf minus Hankel operators, let us recall the definition of Fourier multiplier on $L^p(\mathbb{R})$.

A function $\phi \in L^\infty(\mathbb{R})$ is called a *Fourier multiplier on* $L^p(\mathbb{R})$ if the operator $\mathcal{F}^{-1}\phi \cdot \mathcal{F}$, acting on $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$, extends by continuity to a bounded operator on $L^p(\mathbb{R})$. The set of all Fourier multipliers on $L^p(\mathbb{R})$ is denoted by $\mathcal{M}^p(\mathbb{R})$ and it is a Banach algebra when endowed with the norm

$$\|\phi\|_{\mathcal{M}^p(\mathbb{R})} := \|\mathcal{F}^{-1}\phi \cdot \mathcal{F}\|_{\mathcal{L}(L^p(\mathbb{R}))} \tag{1.1.24}$$

and pointwise multiplication. Further, the set of all Fourier multipliers on $L^2(\mathbb{R})$ coincides with $L^\infty(\mathbb{R})$, i.e.,

$$\mathcal{M}^2(\mathbb{R}) = L^\infty(\mathbb{R}). \tag{1.1.25}$$

We are now in position to present the main objects of this thesis - the Wiener-Hopf plus Hankel operators and the Wiener-Hopf minus Hankel operators. As the names suggest, these operators result from combinations between Wiener-Hopf and Hankel operators. Combinations of such kind appear for the first time in 1979 in the classical work of S. Power [62], where a study of the spectra and essential spectra of Hankel operators was presented by investigating the C^*–algebra generated by Toeplitz and Hankel operators (in the two cases of piecewise continuous symbols and almost periodic symbols). Although in the major part of this thesis we will consider operators acting between L^2 Lebesgue spaces, we will now define Wiener-Hopf plus Hankel operators and Wiener-Hopf minus Hankel operators acting between L^p Lebesgue spaces.

For $\phi \in \mathcal{M}^p(\mathbb{R})$, let W_ϕ and H_ϕ be the *Wiener-Hopf* and *Hankel operators* defined by

$$W_\phi := r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F} \; : \; L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+) \tag{1.1.26}$$

$$H_\phi := r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}J \; : \; L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+), \tag{1.1.27}$$

respectively. The Wiener-Hopf plus Hankel operator is given by

$$(W+H)_\phi := W_\phi + H_\phi \; : \; L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+), \tag{1.1.28}$$

and the Wiener-Hopf minus Hankel operator is defined by

$$(W-H)_\phi := W_\phi - H_\phi \; : \; L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+). \tag{1.1.29}$$

For a Banach space Z, let I_Z represent the *identity operator on* Z. According to (1.1.26), (1.1.27) and (1.1.28), we have

$$(W+H)_\phi = r_+(\mathcal{F}^{-1}\phi \cdot \mathcal{F} + \mathcal{F}^{-1}\phi \cdot \mathcal{F}J) = r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}(I_{L^p_+(\mathbb{R})} + J). \tag{1.1.30}$$

Furthermore, since

$$I_{L^p_+(\mathbb{R})} + J = \ell^e r_+\,, \tag{1.1.31}$$

where ℓ^e denotes the *even extension operator from* $L^p(\mathbb{R}_+)$ *into* $L^p(\mathbb{R})$, we may write the Wiener-Hopf plus Hankel operator as

$$(W+H)_\phi = r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}\ell^e r_+\,. \tag{1.1.32}$$

Combining now (1.1.26), (1.1.27) and (1.1.29), we have

$$(W-H)_\phi = r_+(\mathcal{F}^{-1}\phi \cdot \mathcal{F} - \mathcal{F}^{-1}\phi \cdot \mathcal{F}J) = r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}(I_{L^p_+(\mathbb{R})} - J). \tag{1.1.33}$$

Denoting by ℓ^o the *odd extension operator from* $L^p(\mathbb{R}_+)$ *into* $L^p(\mathbb{R})$, it follows that

$$I_{L^p_+(\mathbb{R})} - J = \ell^o r_+\,. \tag{1.1.34}$$

In this way, we may write the Wiener-Hopf minus Hankel operator as

$$(W-H)_\phi = r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}\ell^o r_+\,. \tag{1.1.35}$$

From (1.1.32) and (1.1.35), we observe that Wiener-Hopf-Hankel operators may be written as convolution type operators with symmetry. Notice that *convolution type operators with symmetry (CTOS)* are operators of the form

$$T = r_+ A l^c : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)\,, \tag{1.1.36}$$

where $A = \mathcal{F}^{-1}\phi \cdot \mathcal{F} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is a translation invariant operator and supposed to be bounded invertible, and ℓ^c denotes *even or odd extension* as a continuous operator from $L^p(\mathbb{R}_+)$ into $L^p(\mathbb{R})$. CTOS may also be considered in Bessel potential spaces. Operators of this type were first studied in [51] and then in [22].

1.2 Regularity properties

Let X and Y be two Banach spaces and consider $T \in \mathcal{L}(X, Y)$. The *kernel* $\operatorname{Ker} T$ and the *image* $\operatorname{Im} T$ of the operator T are defined by

$$\operatorname{Ker} T := \{x \in X : Tx = 0\}, \quad \operatorname{Im} T := \{Tx : x \in X\}. \tag{1.2.37}$$

$\operatorname{Ker} T$ and $\operatorname{Im} T$ are linear subspaces of X and Y, respectively. Moreover, $\operatorname{Ker} T$ is a closed subspace. If $\operatorname{Im} T$ is also a closed subspace, the operator T is said to be *normally solvable*. In this case, the *cokernel of* T is defined as

$$\operatorname{Coker} T := Y/\operatorname{Im} T. \tag{1.2.38}$$

For a normally solvable operator T, the *deficiency numbers* of T are defined as

$$n(T) := \dim \operatorname{Ker} T, \quad d(T) := \dim \operatorname{Coker} T. \tag{1.2.39}$$

Concerning the kernel of T, it holds that $\operatorname{Ker} T$ satisfies (at least) one of the following conditions:

(1) $n(T) = 0$;

(2) $n(T) < \infty$;

(3) $\operatorname{Ker} T$ is complementable in X;

(4) or $\operatorname{Ker} T$ is closed in X.

A similar characterization can be done for the image of T, that is, $\operatorname{Im} T$ satisfies (at least) one of the following conditions:

(i) $d(T) = 0$;

(ii) $d(T) < \infty$;

(iii) $\operatorname{Im} T$ is complementable in Y;

(iv) or $\operatorname{Im} T$ is closed in Y.

Combining these properties concerning to the kernel of T with the properties referring to the image of T, we obtain several classes that are called *regularity classes* [16, 18, 73]. In what follows, we will define some of these regularity classes.

The operator T is said to be a *Fredholm* operator if it is normally solvable and $n(T)$ and $d(T)$ are finite. In this case, the *Fredholm index* of T is defined by

$$\operatorname{Ind} T := n(T) - d(T). \tag{1.2.40}$$

The operator T is said to be a *semi-Fredholm* operator if it is normally solvable and at least one of the deficiency numbers $n(T)$ and $d(T)$ is finite. A semi-Fredholm operator is said to be *n-normal* if $n(T)$ is finite, and *d-normal* if $d(T)$ is finite. In the case where only one of the deficiency numbers is finite, the operator T is said to be a *properly semi--Fredholm* operator. In this case, the operator T is said to be *properly n-normal* if $n(T)$ is finite and $d(T)$ is infinite, and *properly d-normal* if $d(T)$ is finite and $n(T)$ is infinite. For semi-Fredholm operators, the index formula (1.2.40) is also well-defined. In particular, for properly semi-Fredholm operators, we have $\operatorname{Ind} T = -\infty$ if T is properly n-normal and $\operatorname{Ind} T = +\infty$ if T is properly d-normal.

We point out that in German and Russian literature, (semi-)Fredholm operators are often called (semi-)*Noether operators*. This is due to the pioneering work [56] of F. Noether who was the first to discover that singular integral operators with nonvanishing continuous symbols are normally solvable, and have finite kernel and cokernel dimensions.

Still concerning to semi-Fredholm operators, we present next a property about the index of these operators which will be used later on.

Theorem 1.2.1. (cf. [35, §6.7]) *If $T \in \mathcal{L}(X, Y)$ is a Fredholm operator (resp. properly n-normal, properly d-normal), then there exists a number $\delta > 0$ such that, for all operators $M \in \mathcal{L}(X, Y)$ with $\|M\| < \delta$, $A + M \in \mathcal{L}(X, Y)$ is a Fredholm operator (resp. properly n-normal, properly d-normal) and* $\operatorname{Ind}(A + M) = \operatorname{Ind} A$.

We end up this section by introducing the definition of reflexive generalized invertibility. As we will see in a moment, it is possible to characterize the regularity class of all operators T such that $\operatorname{Ker} T$ is a complementable subspace of X and $\operatorname{Im} T$ is a closed and complementable subspace of Y in terms of the reflexive generalized invertibility of T.

Let $T : X \to Y$ be a bounded linear operator acting between Banach spaces. T is said to be *reflexive generalized invertible* if there exists a bounded linear operator $T^- : Y \to X$ such that

$$TT^-T = T \text{ and } T^-TT^- = T^-. \tag{1.2.41}$$

In this case, the operator T^- is referred to as the *reflexive generalized inverse* (or *pseudoinverse*) of T. From the definition of reflexive generalized invertible operator, it follows that a linear bounded one-sided or two-sided invertible operator is also a reflexive generalized invertible operator and one of its reflexive generalized inverses is the one-sided or two-sided inverse, respectively. In general, reflexive generalized inverses are not unique. However, in the case where T is invertible, the reflexive generalized inverses are unique and coincide with the inverse of T.

Finally, and about the characterization (mentioned above) of the regularity class of all operators T such that $\operatorname{Ker} T$ is a complementable subspace of X and $\operatorname{Im} T$ is a closed and complementable subspace of Y, we have the following: T is reflexive generalized invertible if and only if $\operatorname{Ker} T$ is a complementable subspace of X and $\operatorname{Im} T$ is a closed and complementable subspace of Y. From this result, it holds that each Fredholm operator is reflexive generalized invertible.

1.3 Relations between convolution type operators

1.3.1 Relations between bounded linear operators

In what follows, consider $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$ two bounded linear operators acting between Banach spaces.

The operators T and S are said to be *equivalent*, and we will denote this by $T \sim S$, if there are two boundedly invertible linear operators, $E : Y_2 \to X_2$ and $F : X_1 \to Y_1$, such that

$$T = E\, S\, F. \tag{1.3.42}$$

It directly follows from (1.3.42) that if two operators are equivalent, then they belong to the same regularity class. Namely, one of these operators is invertible, one-sided invertible, Fredholm, (properly) n-normal, (properly) d-normal or normally solvable, if and only if the other operator enjoys the same property.

An operator relation that generalizes the operator equivalence relation is the equivalence after extension relation. The operators T and S are said to be *equivalent after extension*, and we will denote this by $T \overset{*}{\sim} S$, if there exist two Banach spaces W and Z such that $T \oplus I_W$ and $S \oplus I_Z$ are equivalent operators, i.e.,

$$\begin{bmatrix} T & 0 \\ 0 & I_W \end{bmatrix} = E \begin{bmatrix} S & 0 \\ 0 & I_Z \end{bmatrix} F, \tag{1.3.43}$$

for invertible bounded linear operators $E : Y_2 \times Z \to X_2 \times W$ and $F : X_1 \times W \to Y_1 \times Z$. As we can easily see, the operator equivalence relation corresponds to the case where the extension spaces W and Z are chosen to be the trivial space (in the equivalence after extension relation). Like in the equivalence case, two equivalent after extension operators belong to the same regularity class.

Another known relation between bounded linear operators is the matricial coupling. We say that T and S are *matricially coupled* if they can be dilated to invertible bounded linear operators

$$\begin{bmatrix} T & T_2 \\ T_1 & T_0 \end{bmatrix} : X_1 \times Y_2 \to X_2 \times Y_1 \,, \tag{1.3.44}$$

$$\begin{bmatrix} S_0 & S_1 \\ S_2 & S \end{bmatrix} : X_2 \times Y_1 \to X_1 \times Y_2 \tag{1.3.45}$$

such that

$$\begin{bmatrix} T & T_2 \\ T_1 & T_0 \end{bmatrix}^{-1} = \begin{bmatrix} S_0 & S_1 \\ S_2 & S \end{bmatrix} . \tag{1.3.46}$$

In [3], H. Bart, I. Gohberg, and M. A. Kaashoek proved that matricial coupling implies equivalence after extension. Later on, in [4], H. Bart and V. E. Tsekanovshii proved that the converse is also true, and therefore, matricial coupling and equivalence after extension amount to the same. That is to say that two operators acting between Banach spaces are equivalent after extension if and only if they are matricially coupled.

Concerning additional operators relations, we find the Δ*–relation* and the Δ*–relation after extension* that were introduced by L. P. Castro and F.-O. Speck in [15, 18]. In regard to the Δ–relation, that can be viewed as a generalization of the equivalence relation, we say that T is Δ*–related* with S if there is a bounded linear operator acting between Banach spaces $T_\Delta : X_{1\Delta} \to X_{2\Delta}$ and two invertible bounded linear operators $E : Y_2 \to X_2 \times X_{2\Delta}$ and $F : X_1 \times X_{1\Delta} \to Y_1$ such that

$$\begin{bmatrix} T & 0 \\ 0 & T_\Delta \end{bmatrix} = ESF. \tag{1.3.47}$$

As we can see here, in general the extension is not made with the identity operator, like in the equivalence after extension relation (cf. (1.3.43)), but with a third new operator T_Δ. Therefore, due to the presence of three operators, T, T_Δ and S, the relation (1.3.47) is called Δ–relation.

In its turn, the Δ–relation after extension appears a generalization of the Δ–relation. Thus, T is said to be Δ*–related after extension with* S if there is a bounded linear operator acting between Banach spaces $T_\Delta : X_{1\Delta} \to X_{2\Delta}$, a Banach space Z and invertible bounded linear operators $E : Y_2 \times Z \to X_2 \times X_{2\Delta}$ and $F : X_1 \times X_{1\Delta} \to Y_1 \times Z$ such that

$$\begin{bmatrix} T & 0 \\ 0 & T_\Delta \end{bmatrix} = E \begin{bmatrix} S & 0 \\ 0 & I_Z \end{bmatrix} F. \tag{1.3.48}$$

From (1.3.47) and (1.3.48) it follows that if we have T being Δ–related with S or T being Δ–related after extension with S, then the transfer of regularity properties can only be guaranteed in one direction, that is, from operator S to operator T, as stated in [18, Theorem 2.1]. It is clear that this restriction occurs only here and in contrast to what happens with the transfer of regularity properties between two equivalent (after extension) operators, where the transfer can be done in both directions.

1.3.2 Factorization of Wiener-Hopf plus Hankel operators

In order to obtain invertibility and semi-Fredholm criteria for Wiener-Hopf plus Hankel operators with almost periodic Fourier symbols, and acting between L^2 Lebesgue spaces, we need a factorization theory for these operators. In this sense, in what follows, we will consider Wiener-Hopf, Hankel and Wiener-Hopf-Hankel operators acting between L^2 Lebesgue spaces. We point out that similar results hold true for operators acting between L^p Lebesgue spaces.

We will start by recalling two well-known relations between Wiener-Hopf and Hankel operators that arise from the Wiener-Hopf and Hankel operator theory.

Proposition 1.3.1. ([12, Proposition 2.10],[75]) *Let* $\phi, \varphi \in L^\infty(\mathbb{R})$. *Then*

$$W_{\phi\varphi} = W_\phi\, \ell_0\, W_\varphi + H_\phi\, \ell_0\, H_{\widetilde{\varphi}}, \tag{1.3.49}$$

$$H_{\phi\varphi} = W_\phi\, \ell_0\, H_\varphi + H_\phi\, \ell_0\, W_{\widetilde{\varphi}}. \tag{1.3.50}$$

Proof. Taking into account that

$$I_{L^2(\mathbb{R})} = P_+ + P_- = P_+ + P_- JJP_- \tag{1.3.51}$$

and

$$P_- J = JP_+\,, \quad JP_- = P_+ J, \tag{1.3.52}$$

it holds

$$I_{L^2(\mathbb{R})} = P_+ + JP_+P_+J = P_+ + JP_+J. \tag{1.3.53}$$

Therefore, it follows

$$\begin{aligned}
W_{\phi\varphi} &= r_+\mathcal{F}^{-1}(\phi\varphi)\cdot\mathcal{F} \\
&= r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}\mathcal{F}^{-1}\varphi\cdot\mathcal{F} \\
&= r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}(P_+ + JP_+J)\mathcal{F}^{-1}\varphi\cdot\mathcal{F} \\
&= r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}P_+\mathcal{F}^{-1}\varphi\cdot\mathcal{F} + r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}JP_+J\mathcal{F}^{-1}\varphi\cdot\mathcal{F} \\
&= r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}\ \ell_0\ r_+\mathcal{F}^{-1}\varphi\cdot\mathcal{F} + r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}J\ \ell_0\ r_+\mathcal{F}^{-1}\widetilde{\varphi}\cdot\mathcal{F}J \\
&= W_\phi\,\ell_0\,W_\varphi + H_\phi\,\ell_0\,H_{\widetilde{\varphi}}
\end{aligned} \tag{1.3.54}$$

and

$$\begin{aligned}
H_{\phi\varphi} &= r_+\mathcal{F}^{-1}(\phi\varphi)\cdot\mathcal{F}J \\
&= r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}\mathcal{F}^{-1}\varphi\cdot\mathcal{F}J \\
&= r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}(P_+ + JP_+J)\mathcal{F}^{-1}\varphi\cdot\mathcal{F}J \\
&= r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}P_+\mathcal{F}^{-1}\varphi\cdot\mathcal{F}J + r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}JP_+J\mathcal{F}^{-1}\varphi\cdot\mathcal{F}J \\
&= r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}\ \ell_0\ r_+\mathcal{F}^{-1}\varphi\cdot\mathcal{F}J + r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}J\ \ell_0\ r_+\mathcal{F}^{-1}\widetilde{\varphi}\cdot\mathcal{F} \\
&= W_\phi\,\ell_0\,H_\varphi + H_\phi\,\ell_0\,W_{\widetilde{\varphi}}.
\end{aligned} \tag{1.3.55}$$

□

Having in mind Proposition 1.3.1 and thinking on the class of Wiener-Hopf plus Hankel operators, it is natural to expect a corresponding formula for this class of operators. In fact, for Wiener-Hopf plus Hankel operators also holds an analogue of formula (1.3.49) like it is stated in the following proposition.

Proposition 1.3.2. *Let $\phi, \varphi \in L^\infty(\mathbb{R})$. Then*

$$(W+H)_{\phi\varphi} = (W+H)_\phi\,\ell_0\,(W+H)_\varphi + H_\phi\,\ell_0\,(W+H)_{\widetilde{\varphi}-\varphi}\,. \tag{1.3.56}$$

Proof. Adding (1.3.49) and (1.3.50), we obtain

$$(W+H)_{\phi\varphi} = W_\phi\,\ell_0\,(W+H)_\varphi + H_\phi\,\ell_0\,(W+H)_{\tilde{\varphi}}. \tag{1.3.57}$$

Adding and subtracting $H_\phi\,\ell_0\,(W+H)_\varphi$ on the right-hand side of the last identity, it holds

$$(W+H)_{\phi\varphi} = (W+H)_\phi\,\ell_0\,(W+H)_\varphi + H_\phi\,\ell_0\,(W+H)_{\tilde{\varphi}-\varphi}. \tag{1.3.58}$$

$\square$

Let $\mathbb{C}_+ := \{z \in \mathbb{C} : \Im m\, z > 0\}$ and $\mathbb{C}_- := \{z \in \mathbb{C} : \Im m\, z < 0\}$. As usual, let $H^\infty(\mathbb{C}_\pm)$ denote the set of all bounded and analytic functions in $\mathbb{C}_\pm$. *Fatou's Theorem* asserts that functions in $H^\infty(\mathbb{C}_\pm)$ have non-tangential limits on $\mathbb{R} = \partial\mathbb{C}_\pm$ almost everywhere. In this sense, let $H^\infty_\pm(\mathbb{R})$ be the set of all functions in $L^\infty(\mathbb{R})$ that are non-tangential limits of elements in $H^\infty(\mathbb{C}_\pm)$. $H^\infty_+(\mathbb{R})$ and $H^\infty_-(\mathbb{R})$ are closed subalgebras of $L^\infty(\mathbb{R})$. For $0 < p < \infty$, $H^p(\mathbb{C}_\pm)$ denote the set of all functions ϕ which are analytic in $\mathbb{C}_\pm$ and such that

$$\sup_{\pm y>0} \int_{\mathbb{R}} |\phi(x+iy)|^p\, dy < \infty\ . \tag{1.3.59}$$

Like in the case of $H^\infty(\mathbb{C}_\pm)$, by Fatou's theorem, it also holds that functions in $H^p(\mathbb{C}_\pm)$ have non-tangential limits almost everywhere on $\mathbb{R}$. The set of all these non-tangential functions is denoted by $H^p_\pm(\mathbb{R})$. For $1 < p < \infty$, $H^p_\pm(\mathbb{R})$ is a closed subspace of $L^p(\mathbb{R})$.

It is well-known that:

$$\text{if } \phi \in H^\infty_+(\mathbb{R}), \text{ then } H_{\tilde{\phi}} = 0; \tag{1.3.60}$$

$$\text{if } \phi \in H^\infty_-(\mathbb{R}), \text{ then } H_\phi = 0. \tag{1.3.61}$$

These two simple facts are very important in the theory of Wiener-Hopf operators. Namely, it is possible to factorize the Wiener-Hopf operator if its Fourier symbol admits a factorization where the left factor belongs to $H^\infty_-(\mathbb{R})$ and the right factor belongs to $H^\infty_+(\mathbb{R})$.

Proposition 1.3.3. [14, §9.5] *If $\varphi_\pm \in H^\infty_\pm(\mathbb{R})$ and $\psi \in L^\infty(\mathbb{R})$, then*

$$W_{\varphi_-\,\psi\,\varphi_+} = W_{\varphi_-}\,\ell_0\,W_\psi\,\ell_0\,W_{\varphi_+}. \tag{1.3.62}$$

Proof. The result is a direct consequence of (1.3.49), (1.3.60) and (1.3.61). □

With a convenient change, it is possible to construct for Wiener-Hopf plus Hankel operators a corresponding result as the previous one for Wiener-Hopf operators. Due to (1.3.56), if we consider $\phi \in H^\infty_-(\mathbb{R})$ or φ being an even function belonging to $L^\infty(\mathbb{R})$, then we obtain a factorization of the Wiener-Hopf plus Hankel operator

$$(W+H)_{\phi\varphi} = (W+H)_\phi \, \ell_0 \, (W+H)_\varphi \, . \tag{1.3.63}$$

This means that in the Wiener-Hopf plus Hankel case we may factorize the operator on the left if the left factor belongs to $H^\infty_-(\mathbb{R})$ and on the right if the right factor is an even function belonging to $L^\infty(\mathbb{R})$.

Proposition 1.3.4. *Let $\phi, \psi, \varphi \in L^\infty(\mathbb{R})$. If $\phi \in H^\infty_-(\mathbb{R})$ and $\varphi = \widetilde{\varphi}$, then the following operator factorization takes place:*

$$\begin{aligned} (W+H)_{\phi\,\psi\,\varphi} &= (W+H)_\phi \, \ell_0 \, (W+H)_\psi \, \ell_0 \, (W+H)_\varphi & (1.3.64)\\ &= W_\phi \, \ell_0 \, (W+H)_\psi \, \ell_0 \, (W+H)_\varphi . & (1.3.65) \end{aligned}$$

Proof. From the hypothesis $\phi \in H^\infty_-(\mathbb{R})$, we may apply the already presented factorization of the Wiener-Hopf plus Hankel operators, see (1.3.63). Thus

$$(W+H)_{\phi\ \psi\varphi} = (W+H)_\phi \, \ell_0 \, (W+H)_{\psi\varphi}. \tag{1.3.66}$$

In addition, since $\varphi = \widetilde{\varphi}$, it also follows from (1.3.63) that

$$(W+H)_{\psi\varphi} = (W+H)_\psi \, \ell_0 \, (W+H)_\varphi. \tag{1.3.67}$$

From (1.3.66) and (1.3.67), we have that

$$(W+H)_{\phi\,\psi\,\varphi} = (W+H)_\phi \, \ell_0 \, (W+H)_\psi \, \ell_0 \, (W+H)_\varphi. \tag{1.3.68}$$

Since $\phi \in H^\infty_-(\mathbb{R})$, we have $H_\phi = 0$ and consequently $(W+H)_\phi = W_\phi$. Finally, it follows from (1.3.68) that

$$(W+H)_{\phi\,\psi\,\varphi} = W_\phi \, \ell_0 \, (W+H)_\psi \, \ell_0 \, (W+H)_\varphi. \tag{1.3.69}$$

□

From (1.3.63) we have that if the symbol of the Wiener-Hopf plus Hankel operator is factorized in such a way that the right factor is an even function belonging to $L^\infty(\mathbb{R})$, this leads to a factorization of the Wiener-Hopf plus Hankel operator, where a Wiener Hopf plus Hankel operator with an even symbol appears. Due to the multiplicative relation for Wiener-Hopf plus Hankel operators (1.3.63), we conclude that the Wiener Hopf plus Hankel operator with an even symbol is an invertible operator. So, we end this section with this result. Here and in what follows, we will use the notation $\mathcal{G}B$ for the group of all invertible elements of a Banach algebra B. We point out that $\mathcal{G}B$ is an open subset of B and is a group with respect to the multiplication on B.

Proposition 1.3.5. *If $\phi_e \in \mathcal{G}L^\infty(\mathbb{R})$ and $\widetilde{\phi_e} = \phi_e$, then $(W+H)_{\phi_e}$ is invertible and its inverse is the operator*

$$\ell_0\,(W+H)_{\phi_e^{-1}}\,\ell_0 : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R}). \tag{1.3.70}$$

Proof. On the one hand, we have

$$(W+H)_{\phi_e\cdot\phi_e^{-1}}\,\ell_0 = (W+H)_1\,\ell_0 = W_1\,\ell_0 = I_{L^2(\mathbb{R}_+)}. \tag{1.3.71}$$

On the other hand, since $\phi_e \in \mathcal{G}L^\infty(\mathbb{R}))$ and $\widetilde{\phi_e} = \phi_e$, then $\widetilde{\phi_e^{-1}} = \phi_e^{-1}$ and therefore we may apply the factorization of the Wiener-Hopf plus Hankel operators (1.3.63). So we have

$$(W+H)_{\phi_e\cdot\phi_e^{-1}} = (W+H)_{\phi_e}\,\ell_0\,(W+H)_{\phi_e^{-1}}. \tag{1.3.72}$$

Thus, combining (1.3.71) and (1.3.72), we get that

$$(W+H)_{\phi_e}\,\ell_0\,(W+H)_{\phi_e^{-1}}\,\ell_0 = I_{L^2(\mathbb{R}_+)}. \tag{1.3.73}$$

In the same way, we obtain that

$$\ell_0\,(W+H)_{\phi_e^{-1}}\,\ell_0\,(W+H)_{\phi_e} = I_{L^2_+(\mathbb{R})}. \tag{1.3.74}$$

Therefore, (1.3.73)–(1.3.74) show that $(W+H)_{\phi_e}$ is invertible and its inverse is

$$\ell_0\,(W+H)_{\phi_e^{-1}}\,\ell_0 : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R}). \tag{1.3.75}$$

$\square$

A similar result of Proposition 1.3.5 for Toeplitz plus Hankel operators is known in [5, Corollary 2.3].

Remark 1.3.6. We would like to stress that for Wiener-Hopf minus Hankel operators we obtain an analogue of Proposition 1.3.2 and due to this we reach a similar result of Proposition 1.3.4. That is, for $\phi, \varphi \in L^\infty(\mathbb{R})$, we have

$$(W-H)_{\phi\varphi} = (W-H)_\phi \, \ell_0 \, (W-H)_\varphi + H_\phi \, \ell_0 \, (W-H)_{\varphi-\widetilde{\varphi}} \, , \tag{1.3.76}$$

and, considering $\phi, \varphi \in L^\infty(\mathbb{R})$ such that $\phi \in H^\infty_-(\mathbb{R})$ or $\varphi = \widetilde{\varphi}$, we obtain a similar formula as in (1.3.63) for the factorization of the Wiener-Hopf minus Hankel operator:

$$(W-H)_{\phi\varphi} = (W-H)_\phi \, \ell_0 \, (W-H)_\varphi \, . \tag{1.3.77}$$

From here, it follows that, for $\phi, \psi, \varphi \in L^\infty(\mathbb{R})$, if $\phi \in H^\infty_-(\mathbb{R})$ and $\varphi = \widetilde{\varphi}$, then

$$\begin{aligned} (W-H)_{\phi\,\psi\,\varphi} &= (W-H)_\phi \, \ell_0 \, (W-H)_\psi \, \ell_0 \, (W-H)_\varphi & (1.3.78) \\ &= W_\phi \, \ell_0 \, (W-H)_\psi \, \ell_0 \, (W-H)_\varphi. & (1.3.79) \end{aligned}$$

1.3.3 Relation between Wiener-Hopf-Hankel operators and Wiener-Hopf operators

Following the spirit of the *Gohberg-Krupnik-Litvinchuk identity* (cf. [15, 43, 46, 48, 64]), we will describe a relation between Wiener-Hopf plus Hankel operators and Wiener-Hopf operators. Such relation is also based on Wiener-Hopf minus Hankel and paired operators, and it will be very important in the obtainment of the Sarason's type theorem, the Duduchava-Saginashvili's type theorem, and the invertibility and semi-Fredholm criteria for Wiener-Hopf-Hankel operators with piecewise almost periodic symbols and for matrix Wiener-Hopf-Hankel operators with good Hausdorff sets (to be presented later).

The results presented ahead are obtained for the general case of Wiener-Hopf-Hankel operators acting in L^p Lebesgue spaces. In particular, these results hold true in the case where we consider Wiener-Hopf-Hankel operators acting in L^2 Lebesgue spaces. Please recall that, in this particular case, we have $\mathcal{M}^2(\mathbb{R}) = L^\infty(\mathbb{R})$.

Lemma 1.3.7. (cf. [15, Example 5.2.9], [18]) *Let $\phi \in \mathcal{GM}^p(\mathbb{R})$. The Wiener-Hopf plus Hankel operator*

$$(W+H)_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+) \tag{1.3.80}$$

is Δ–related after extension with the Wiener-Hopf operator

$$W_{\phi\widetilde{\phi^{-1}}} : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+). \tag{1.3.81}$$

Proof. Extending $(W+H)_\phi$ on the left by the zero extension operator $\ell_0 : L^p(\mathbb{R}_+) \to L^p_+(\mathbb{R})$, we obtain

$$(W+H)_\phi \sim \ell_0(W+H)_\phi : L^p_+(\mathbb{R}) \to L^p_+(\mathbb{R}). \tag{1.3.82}$$

After this, we will extend

$$\ell_0(W+H)_\phi = P_+\mathcal{F}^{-1}(\phi\cdot + \phi\cdot J)\mathcal{F}_{|P_+L^p(\mathbb{R})} \tag{1.3.83}$$

to the full $L^p(\mathbb{R})$ space by using the identity in $L^p_-(\mathbb{R})$. Next we will extend the obtained operator to $[L^p(\mathbb{R})]^2$ with the help of the auxiliar paired operator

$$\mathcal{T}_\phi = \mathcal{F}^{-1}(\phi\cdot - \phi\cdot J)\mathcal{F}P_+ + P_- : L^p(\mathbb{R}) \to L^p(\mathbb{R})\,. \tag{1.3.84}$$

Altogether, we have

$$\left[\begin{array}{cc|c} \ell_0(W+H)_\phi & 0 & 0 \\ 0 & I_{P_-L^p(\mathbb{R})} & 0 \\ \hline 0 & 0 & \mathcal{T}_\phi \end{array}\right] = E_1\mathcal{W}_\Phi F_1 \tag{1.3.85}$$

with

$$E_1 = \frac{1}{2}\begin{bmatrix} I_{L^p(\mathbb{R})} & J \\ I_{L^p(\mathbb{R})} & -J \end{bmatrix}, \tag{1.3.86}$$

$$F_1 = \begin{bmatrix} I_{L^p(\mathbb{R})} & I_{L^p(\mathbb{R})} \\ J & -J \end{bmatrix} \begin{bmatrix} I_{L^p(\mathbb{R})} - P_-\mathcal{F}^{-1}(\phi \cdot -\phi \cdot J)\mathcal{F}P_+ & 0 \\ 0 & I_{L^p(\mathbb{R})} \end{bmatrix}, \tag{1.3.87}$$

$$\begin{aligned} \mathcal{W}_\Phi &= \begin{bmatrix} \mathcal{F}^{-1}\phi \cdot \mathcal{F} & 0 \\ \mathcal{F}^{-1}\widetilde{\phi} \cdot \mathcal{F} & 1 \end{bmatrix} P_+ + \begin{bmatrix} 1 & \mathcal{F}^{-1}\phi \cdot \mathcal{F} \\ 0 & \mathcal{F}^{-1}\widetilde{\phi} \cdot \mathcal{F} \end{bmatrix} P_- \\ &= \begin{bmatrix} 1 & \mathcal{F}^{-1}\phi \cdot \mathcal{F} \\ 0 & \mathcal{F}^{-1}\widetilde{\phi} \cdot \mathcal{F} \end{bmatrix} (\mathcal{F}^{-1}\Psi \cdot \mathcal{F}P_+ + P_-) \\ &= \begin{bmatrix} 1 & \mathcal{F}^{-1}\phi \cdot \mathcal{F} \\ 0 & \mathcal{F}^{-1}\widetilde{\phi} \cdot \mathcal{F} \end{bmatrix} (P_+\mathcal{F}^{-1}\Psi \cdot \mathcal{F}P_+ + P_-)(I_{[L^p(\mathbb{R})]^2} + P_-\mathcal{F}^{-1}\Psi \cdot \mathcal{F}P_+), \end{aligned} \tag{1.3.88}$$

where in the last definition of operator $\mathcal{W}_\Phi$ we are using $P_\pm$ defined in $[L^p(\mathbb{R})]^2$ and

$$\Psi := \begin{bmatrix} 0 & -\phi\widetilde{\phi^{-1}} \\ 1 & \widetilde{\phi^{-1}} \end{bmatrix}. \tag{1.3.89}$$

We point out that the paired operator

$$I_{[L^p(\mathbb{R})]^2} + P_-\mathcal{F}^{-1}\Psi \cdot \mathcal{F}P_+ : [L^p(\mathbb{R})]^2 \to [L^p(\mathbb{R})]^2 \tag{1.3.90}$$

used above is an invertible operator with inverse given by

$$I_{[L^p(\mathbb{R})]^2} - P_-\mathcal{F}^{-1}\Psi \cdot \mathcal{F}P_+ : [L^p(\mathbb{R})]^2 \to [L^p(\mathbb{R})]^2. \tag{1.3.91}$$

Therefore, we have just explicitly demonstrated that $(W+H)_\phi$ is Δ–related after extension with

$$W_\Psi := r_+\mathcal{F}^{-1}\Psi \cdot \mathcal{F} : [L^p_+(\mathbb{R})]^2 \to [L^p(\mathbb{R}_+)]^2\,. \tag{1.3.92}$$

Furthermore, we have

$$\begin{bmatrix} W_{\phi\widetilde{\phi^{-1}}}\ell_0 & 0 \\ 0 & I_{L^p(\mathbb{R}_+)} \end{bmatrix} = W_\Psi \ell_0 r_+ \mathcal{F}^{-1} \begin{bmatrix} \widetilde{\phi^{-1}} & 1 \\ -1 & 0 \end{bmatrix} \mathcal{F}\ell_0 : [L^p(\mathbb{R}_+)]^2 \to [L^p(\mathbb{R}_+)]^2 \tag{1.3.93}$$

which shows an explicit equivalence after extension relation between $W_{\phi\widetilde{\phi^{-1}}}$ and W_Ψ. This together with the Δ–relation after extension between $(W+H)_\phi$ and W_Ψ concludes the proof. □

Corollary 1.3.8. *Let $\phi \in \mathcal{GM}^p(\mathbb{R})$. If the Wiener-Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ is invertible, left--invertible, right-invertible, Fredholm, n-normal, d-normal or normally solvable, then the Wiener-Hopf plus Hankel operator $(W+H)_\phi$ has the same property as $W_{\phi\widetilde{\phi^{-1}}}$.*

Proof. The result is a direct consequence of the Δ–relation after extension between the Wiener-Hopf plus Hankel operator $(W+H)_\phi$ and the Wiener-Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ presented in Lemma 1.3.7, which ensures that:

(i)

$$\operatorname{Im}\begin{bmatrix}(W+H)_\phi & 0\\ 0 & \mathcal{T}_\phi\end{bmatrix} \text{ is closed if and only if } \operatorname{Im} W_{\phi\widetilde{\phi^{-1}}} \text{ is closed;} \tag{1.3.94}$$

(ii)

$$(L^p(\mathbb{R}_+)\times L^p(\mathbb{R}))\setminus \overline{\operatorname{Im}\begin{bmatrix}(W+H)_\phi & 0\\ 0 & \mathcal{T}_\phi\end{bmatrix}} \simeq L^p(\mathbb{R}_+)\setminus \overline{\operatorname{Im} W_{\phi\widetilde{\phi^{-1}}}}\,; \tag{1.3.95}$$

(iii)

$$\operatorname{Ker}\begin{bmatrix}(W+H)_\phi & 0\\ 0 & \mathcal{T}_\phi\end{bmatrix} \simeq \operatorname{Ker} W_{\phi\widetilde{\phi^{-1}}}\,. \tag{1.3.96}$$

□

Since we are also interested in the study of the Fredholm property of the Wiener--Hopf minus Hankel operators, we are also looking for a result that allow us the transfer of regularity properties from the Wiener-Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ to the Wiener-Hopf minus Hankel operator $(W-H)_\phi$, that is, a similar result of Corollary 1.3.8 for the Wiener-Hopf minus Hankel case. In fact, a result of such kind is possible to obtain if we take into account the following operators relation.

Proposition 1.3.9. *Let $\phi \in \mathcal{GM}^p(\mathbb{R})$. The operator*

$$\mathcal{T}_\phi = \mathcal{F}^{-1}(\phi \cdot -\phi \cdot J)\mathcal{F}P_+ + P_- : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \tag{1.3.97}$$

is equivalent after extension to the Wiener-Hopf minus Hankel operator

$$(W - H)_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+). \tag{1.3.98}$$

Proof. We start by observing that

$$\mathcal{T}_\phi = \mathcal{F}^{-1}(\phi \cdot -\phi \cdot J)\mathcal{F}P_+ + P_- : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \tag{1.3.99}$$

and

$$P_+\mathcal{F}^{-1}(\phi \cdot -\phi \cdot J)\mathcal{F}P_+ + P_- : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \tag{1.3.100}$$

are equivalent operators:

$$\mathcal{T}_\phi = \Big(P_+\mathcal{F}^{-1}(\phi \cdot -\phi \cdot J)\mathcal{F}P_+ + P_-\Big)\Big(I_{L^p(\mathbb{R})} + P_-\mathcal{F}^{-1}(\phi \cdot -\phi \cdot J)\mathcal{F}P_+\Big). \tag{1.3.101}$$

In fact, this is the case because the operator

$$I_{L^p(\mathbb{R})} + P_-\mathcal{F}^{-1}(\phi \cdot -\phi \cdot J)\mathcal{F}P_+ : L^p(\mathbb{R}) \to L^p(\mathbb{R}) \tag{1.3.102}$$

used above is an invertible operator with inverse given by

$$I_{L^p(\mathbb{R})} - P_-\mathcal{F}^{-1}(\phi \cdot -\phi \cdot J)\mathcal{F}P_+ : L^p(\mathbb{R}) \to L^p(\mathbb{R}). \tag{1.3.103}$$

Attending now to the direct sum $L^p(\mathbb{R}) = L^p_+(\mathbb{R}) \oplus L^p_-(\mathbb{R})$, we may write the operator $P_+\mathcal{F}^{-1}(\phi \cdot -\phi \cdot J)\mathcal{F}P_+ + P_- : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ in the matrix form

$$\begin{bmatrix} P_+\mathcal{F}^{-1}(\phi \cdot -\phi \cdot J)\mathcal{F}P_+ & 0 \\ 0 & P_- \end{bmatrix} : L^p_+(\mathbb{R}) \times L^p_-(\mathbb{R}) \to L^p_+(\mathbb{R}) \times L^p_-(\mathbb{R}). \tag{1.3.104}$$

Using $P_- = I_{P_-L^p(\mathbb{R})} : L^p_-(\mathbb{R}) \to L^p_-(\mathbb{R})$, it follows that

$$P_+\mathcal{F}^{-1}(\phi \cdot -\phi \cdot J)\mathcal{F}P_+ : L^p_+(\mathbb{R}) \to L^p_+(\mathbb{R}) \tag{1.3.105}$$

is equivalent after extension to $P_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+ + P_- : L^p(\mathbb{R}) \to L^p(\mathbb{R})$. Recalling that $P_+ = \ell_0 r_+$, and taking into consideration the space where the operator $P_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+$ is acting, we have

$$P_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+ = \ell_0 r_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F} : L^p_+(\mathbb{R}) \to L^p_+(\mathbb{R}). \tag{1.3.106}$$

Finally, due to the fact that $\ell_0 : L^p(\mathbb{R}_+) \to L^p_+(\mathbb{R})$ is invertible, it follows that

$$\ell_0 r_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F} : L^p_+(\mathbb{R}) \to L^p_+(\mathbb{R}) \tag{1.3.107}$$

is equivalent to

$$r_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F} = (W - H)_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+). \tag{1.3.108}$$

Altogether, it holds

$$\mathcal{T}_\phi \overset{*}{\sim} (W - H)_\phi \,, \tag{1.3.109}$$

which completes the proof. □

After settling the equivalence after extension between the operators $\mathcal{T}_\phi$ and $(W-H)_\phi$, the transfer of regularity properties from the Wiener-Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ to the Wiener--Hopf minus Hankel operator $(W-H)_\phi$ follows as a corollary of Lemma 1.3.7.

Corollary 1.3.10. *Let $\phi \in \mathcal{GM}^p(\mathbb{R})$. If the Wiener-Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ is invertible, left-invertible, right-invertible, Fredholm, n-normal, d-normal or normally solvable, then the Wiener-Hopf minus Hankel operator $(W-H)_\phi$ has the same property as $W_{\phi\widetilde{\phi^{-1}}}$.*

Proof. The assertion follows directly from the Δ–relation presented in Lemma 1.3.7 and from Proposition 1.3.9. □

1.3.4 Relation between Wiener-Hopf-Hankel operators and Toeplitz-Hankel operators

A relation between Wiener-Hopf-Hankel operators and Toeplitz-Hankel operators appears in the obtainment of some results in Subsections 4.1.3 and 5.2.1. Since there we are dealing with Wiener-Hopf-Hankel operators acting between L^2 Lebesgue spaces, in what follows we will consider operators acting in L^2 Lebesgue spaces and H^2 Hardy spaces.

Consider the *Cauchy singular integral operator* S *on* $L^2(\mathbb{R})$ given by

$$(S\varphi)(\xi) := \frac{1}{\pi i}\int_{\mathbb{R}} \frac{\varphi(x)}{x-\xi}\, dx, \ \xi \in \mathbb{R}, \tag{1.3.110}$$

and the *orthogonal projection of* $L^2(\mathbb{R})$ *onto* $H^2_+(\mathbb{R})$

$$P := \frac{1}{2}\big(I_{L^2_+(\mathbb{R})} + S\big) : L^2(\mathbb{R}) \to L^2(\mathbb{R}). \tag{1.3.111}$$

With the help of Fourier transformations, it is possible to relate P and P_+ as follows:

$$P = \mathcal{F} P_+ \mathcal{F}^{-1}. \tag{1.3.112}$$

Let $\mathbb{D}_+ := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{D}_- := \{z \in \mathbb{C} : |z| \geq 1\} \cup \{\infty\}$. Analogously to the definition of $H^\infty(\mathbb{C}_\pm)$ presented in Section 1.3.2, $H^\infty(\mathbb{D}_\pm)$ denotes the set of all bounded and analytic functions in $\mathbb{D}_\pm$. Considering $1 \leq p < \infty$, $H^p(\mathbb{D}_+)$ denotes the set of all functions ϕ which are analytic in $\mathbb{D}_+$ and such that

$$\sup_{r\in(0,1)} \int_0^{2\pi} |\phi(re^{i\theta})|^p \, d\theta < \infty, \tag{1.3.113}$$

and $H^p(\mathbb{D}_-)$ denotes the set of all functions $\phi(z)$ $(z \in \mathbb{D}_-)$ for which $\phi(\frac{1}{z})$ is a function in $H^p(\mathbb{D}_+)$. For $1 \leq p \leq \infty$ and similarly to what happens in the case of $H^p(\mathbb{C}_\pm)$, by a theorem of Fatou, we have that functions in $H^p(\mathbb{D}_\pm)$ have non-tangential limits almost everywhere on the unit circle $\mathbb{T}$ $(\mathbb{T} = \partial\mathbb{D}_\pm)$. In this way, $H^p_\pm(\mathbb{T})$ represents the set of all functions on $\mathbb{T}$ that are non-tangential limits of elements in $H^p(\mathbb{D}_\pm)$. $H^p_\pm(\mathbb{T})$ are closed subspaces of $L^p(\mathbb{T})$.

Let $S_\mathbb{T}$ be the *Cauchy singular integral operator on* $L^2(\mathbb{T})$ defined by

$$(S_\mathbb{T}\varphi)(t) := \frac{1}{\pi i}\int_{\mathbb{T}} \frac{\varphi(\tau)}{\tau - t}\, d\tau, \ t \in \mathbb{T}. \tag{1.3.114}$$

Consider also the *orthogonal projection of* $L^2(\mathbb{T})$ *onto* $H^2_+(\mathbb{T})$,

$$P_\mathbb{T} := \frac{1}{2}(I_{L^2(\mathbb{T})} + S_\mathbb{T}) : L^2(\mathbb{T}) \to L^2(\mathbb{T}). \tag{1.3.115}$$

For any $\nu \in L^\infty(\mathbb{T})$, we will consider Toeplitz plus Hankel operators defined by

$$(T+H)_\nu := P_{\mathbb{T}}\nu P_{\mathbb{T}} + P_{\mathbb{T}}\nu J_{\mathbb{T}} P_{\mathbb{T}} : H^2_+(\mathbb{T}) \to H^2_+(\mathbb{T}) \tag{1.3.116}$$

and Toeplitz minus Hankel operators defined by

$$(T-H)_\nu := P_{\mathbb{T}}\nu P_{\mathbb{T}} - P_{\mathbb{T}}\nu J_{\mathbb{T}} P_{\mathbb{T}} : H^2_+(\mathbb{T}) \to H^2_+(\mathbb{T}), \tag{1.3.117}$$

where $J_{\mathbb{T}} : \theta(t) \to \frac{1}{t}\theta\left(\frac{1}{t}\right)$, $t \in \mathbb{T}$, is the *reflection operator on* $\mathbb{T}$.

The aim of this subsection is the obtainment of a relation between Wiener-Hopf-Hankel operators acting in L^2 Lebesgue spaces defined on the real line and Toeplitz-Hankel operators acting in H^2 Hardy spaces defined on the unit circle. To obtain such relation, we will need to use some isomorphisms that allow us to pass from the real line to the unit circle and vice-versa. Thus, let B_0 be an isometric isomorphism from $L^\infty(\mathbb{R})$ onto $L^\infty(\mathbb{T})$ (as well as from $H^\infty_+(\mathbb{R})$ onto $H^\infty_+(\mathbb{T})$) defined by

$$(B_0\phi)(t) := \phi\left(i\frac{1+t}{1-t}\right),\ t \in \mathbb{T}\backslash\{1\} \tag{1.3.118}$$

and with inverse

$$(B_0^{-1}\psi)(x) := \psi\left(\frac{x-i}{x+i}\right),\ x \in \mathbb{R}. \tag{1.3.119}$$

Consider also the isometric isomorphism of $L^2(\mathbb{T})$ onto $L^2(\mathbb{R})$ (as well as of $H^2_+(\mathbb{T})$ onto $H^2_+(\mathbb{R})$) given by

$$(B\varphi)(x) := \frac{\sqrt{2}}{x+i}\varphi\left(\frac{x-i}{x+i}\right),\ x \in \mathbb{R}. \tag{1.3.120}$$

The inverse of B is given by

$$(B^{-1}\psi)(t) := \frac{i\sqrt{2}}{1-t}\psi\left(i\frac{1+t}{1-t}\right),\ t \in \mathbb{T}\backslash\{1\}. \tag{1.3.121}$$

Additionally, it is also useful to observe that for any $\phi \in L^\infty(\mathbb{R})$ it holds

$$B^{-1}\phi \cdot B = (B_0\phi) I_{L^2(\mathbb{R})}. \tag{1.3.122}$$

Using now the isomorphism B and its inverse, we see how orthogonal projections $P_{\mathbb{T}}$ and P, and reflection operators $J_{\mathbb{T}}$ and J can be related:

$$P_{\mathbb{T}} = B^{-1}PB, \tag{1.3.123}$$

$$JB = -BJ_{\mathbb{T}}. \tag{1.3.124}$$

After all this background on spaces and operators, we are now in position to present how Wiener-Hopf-Hankel operators relate with Toeplitz-Hankel operators.

Lemma 1.3.11. *Let $\phi \in L^\infty(\mathbb{R})$. The Wiener-Hopf plus Hankel operator*

$$(W+H)_\phi \; : \; L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+) \tag{1.3.125}$$

and the Toeplitz minus Hankel operator

$$(T-H)_{B_0\phi} \; : \; H^2_+(\mathbb{T}) \to H^2_+(\mathbb{T}) \tag{1.3.126}$$

are equivalent operators.

Proof. According to (1.1.30), we have that

$$(W+H)_\phi = r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}(I_{L^2_+(\mathbb{R})} + J) : L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+). \tag{1.3.127}$$

From this it follows that

$$\begin{aligned}(W+H)_\phi \sim \ell_0\,(W+H)_\phi &= \ell_0 r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}(I_{L^2_+(\mathbb{R})} + J)\\ &= P_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}(I_{L^2_+(\mathbb{R})} + J)P_+ : L^2_+(\mathbb{R}) \to L^2_+(\mathbb{R}).\end{aligned} \tag{1.3.128}$$

More than this, we have

$$\begin{aligned}P_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}(I_{L^2_+(\mathbb{R})} + J)P_+ \sim \mathcal{F}P_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}(I_{L^2_+(\mathbb{R})} + J)P_+\mathcal{F}^{-1}&\\ : \mathcal{F}L^2_+(\mathbb{R}) \to \mathcal{F}L^2_+(\mathbb{R}),&\end{aligned} \tag{1.3.129}$$

which, due to the fact that $\mathcal{F}L^2_+(\mathbb{R}) = H^2_+(\mathbb{R})$, leads to

$$(W+H)_\phi \sim \mathcal{F}P_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}(I_{L^2_+(\mathbb{R})} + J)P_+\mathcal{F}^{-1} : H^2_+(\mathbb{R}) \to H^2_+(\mathbb{R}). \tag{1.3.130}$$

Thus, taking into account that $J\mathcal{F} = \mathcal{F}J$, (1.3.130) yields that

$$(W+H)_\phi \sim P\phi \cdot (I_{L^2_+(\mathbb{R})} + J)P : H^2_+(\mathbb{R}) \to H^2_+(\mathbb{R}). \tag{1.3.131}$$

Consequently, one obtains

$$\begin{aligned}&(W+H)_\phi \sim B^{-1}P\phi \cdot (I_{L^2_+(\mathbb{R})} + J)PB =\\ &\qquad P_{\mathbb{T}}(B^{-1}\phi \cdot B)P_{\mathbb{T}} + P_{\mathbb{T}}(B^{-1}\phi \cdot JB)P_{\mathbb{T}} : H^2_+(\mathbb{T}) \to H^2_+(\mathbb{T}),\end{aligned} \tag{1.3.132}$$

having in consideration (1.3.123). Now we are able to rewrite the last operator in (1.3.132) as

$$(T-H)_{B_0\phi} = P_{\mathbb{T}}(B_0\phi)P_{\mathbb{T}} - P_{\mathbb{T}}(B_0\phi)J_{\mathbb{T}}P_{\mathbb{T}} : H^2_+(\mathbb{T}) \to H^2_+(\mathbb{T}), \tag{1.3.133}$$

just by observing that it holds (1.3.122) and (1.3.124). □

Corollary 1.3.12. *The operators* $(W+H)_\phi$ *and* $(T-H)_{B_0\phi}$ *have the same regularity properties.*

Proof. The statement is a direct consequence of the equivalence relation presented in Lemma 1.3.11. □

Similarly, one can also relate Wiener-Hopf minus Hankel operators with Toeplitz plus Hankel operators.

Lemma 1.3.13. *Let* $\phi \in L^\infty(\mathbb{R})$. *The Wiener-Hopf minus Hankel operator*

$$(W-H)_\phi \;:\; L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+) \tag{1.3.134}$$

and the Toeplitz plus Hankel operator

$$(T+H)_{B_0\phi} \;:\; H^2_+(\mathbb{T}) \to H^2_+(\mathbb{T}) \tag{1.3.135}$$

are equivalent operators.

Proof. Similar to the proof of Theorem 1.3.11. □

Corollary 1.3.14. *The operators* $(W-H)_\phi$ *and* $(T+H)_{B_0\phi}$ *have the same regularity properties.*

Proof. The result follows directly from the equivalence relation stated in Lemma 1.3.13. □

1.4 Necessary conditions for the semi-Fredholm property

Let X be a Banach algebra with the unit element e. The *spectrum of an element* $\phi \in X$ is defined by

$$sp_X\,\phi := \{\lambda \in \mathbb{C} : \phi - \lambda e \notin \mathcal{G}X\}. \tag{1.4.136}$$

In the particular case where $X = L^\infty(\mathbb{R})$, the spectrum of $\phi \in L^\infty(\mathbb{R})$ is called the *essential range* of ϕ and is denoted by $\mathcal{R}(\phi)$. Therefore, we have

$$\mathcal{R}(\phi) := sp_{L^\infty(\mathbb{R})}\, \phi = \{\lambda \in \mathbb{C} : \phi - \lambda \notin \mathcal{G}L^\infty(\mathbb{R})\}. \tag{1.4.137}$$

Thinking now on the case of operators, the *spectrum of an operator* $T \in \mathcal{L}(X)$ is given by $sp_{\mathcal{L}(X)}T$, i.e.,

$$sp\, T := \{\lambda \in \mathbb{C} : T - \lambda I_X \notin \mathcal{G}\mathcal{L}(X)\}. \tag{1.4.138}$$

Recall that, being $\mathcal{L}(X)$ the set of all bounded linear operators acting in the Banach space X, $\mathcal{L}(X)$ is a unital Banach algebra when endowed with the natural algebraic operations and with the operator norm, and it has as unit element the identity operator I_X. As about the spectrum of operators and related with the Fredholm property, we have the *essential spectrum of an operator* $T \in \mathcal{L}(X)$ defined by

$$sp_{ess}T := \{\lambda \in \mathbb{C} : T - \lambda I_X \text{ is not a Fredholm operator}\}. \tag{1.4.139}$$

From the definitions of spectrum and essential spectrum of an operator, it follows that $sp_{ess}T$ is a compact nonempty subset of $sp\, T$.

Looking now at the operators under study, and considering the general case of bounded linear Wiener-Hopf-Hankel operators acting between L^p Lebesgue spaces, $(W \pm H)_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$, we have

$$sp\,(W\pm H)_\phi = \{\lambda \in \mathbb{C} : \ell_0(W\pm H)_\phi - \lambda I_{L^p_+(\mathbb{R})} \notin \mathcal{G}\mathcal{L}(L^p_+(\mathbb{R}))\}\,, \tag{1.4.140}$$

$$sp_{ess}\,(W\pm H)_\phi = \{\lambda \in \mathbb{C} : \ell_0(W\pm H)_\phi - \lambda I_{L^p_+(\mathbb{R})} \text{ is not a Fredholm operator}\}\,, \tag{1.4.141}$$

as the spectrum and essential spectrum of $(W\pm H)_\phi$, respectively. Notice that, since the operators are acting between different Banach spaces, we need to introduce the zero extension operator in the definitions presented above for the spectrum and essential spectrum of $(W\pm H)_\phi$.

Theorem 1.4.1. *If $\phi \in M^p(\mathbb{R})$ and the operator*

$$(W+H)_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+) \tag{1.4.142}$$

is n-normal or d-normal, then $\phi \in \mathcal{G}M^p(\mathbb{R})$. *Consequently,*

$$\mathcal{R}(\phi) \subset sp_{ess}(W+H)_\phi \subset sp\ (W+H)_\phi. \tag{1.4.143}$$

Proof. Suppose that $(W+H)_\phi$ is a n-normal operator. Then, by Corollary 1.3.12, $(T-H)_{B_0\phi}$ is also a n-normal operator. Due to the n-normality of $(T-H)_{B_0\phi}$, there exist a (finite-rank) projection K onto $\operatorname{Ker}(T-H)_{B_0\phi}$ and a $\delta > 0$ such that

$$\|(T-H)_{B_0\phi} f\|_{H^p_+(\mathbb{T})} + \|Kf\|_{H^p_+(\mathbb{T})} \geq \delta \|f\|_{H^p_+(\mathbb{T})} \tag{1.4.144}$$

for all $f \in H^p_+(\mathbb{T})$. Considering $P_\mathbb{T} f$ instead of f and taking into account the inequality

$$\|P_\mathbb{T} f\|_{L^p(\mathbb{T})} \geq \|f\|_{L^p(\mathbb{T})} - \|(I_{L^p(\mathbb{T})} - P_\mathbb{T})f\|_{L^p(\mathbb{T})}, \tag{1.4.145}$$

it follows that

$$\|(T-H)_{B_0\phi} f\|_{L^p(\mathbb{T})} + \|KP_\mathbb{T} f\|_{L^p(\mathbb{T})} + \delta\|(I_{L^p(\mathbb{T})} - P_\mathbb{T})f\|_{L^p(\mathbb{T})} \geq \delta\|f\|_{L^p(\mathbb{T})} \tag{1.4.146}$$

for all $f \in L^p(\mathbb{T})$. Let us now introduce, for $n \in \mathbb{Z}$, the isometries

$$U^n : L^p(\mathbb{T}) \to L^p(\mathbb{T}), \quad U^n f(t) = t^n f(t). \tag{1.4.147}$$

Replacing f by $U^n f$ in (1.4.146), we obtain

$$\begin{aligned} &\|(T-H)_{B_0\phi} U^n f\|_{L^p(\mathbb{T})} + \|KP_\mathbb{T} U^n f\|_{L^p(\mathbb{T})} + \\ &\qquad \delta\|(I_{L^p(\mathbb{T})} - P_\mathbb{T})U^n f\|_{L^p(\mathbb{T})} \geq \delta\|U^n f\|_{L^p(\mathbb{T})} \end{aligned} \tag{1.4.148}$$

for all $f \in L^p(\mathbb{T})$, and since $U^{\pm n}$ are isometries, it follows

$$\begin{aligned} &\|U^{-n}(T-H)_{B_0\phi} U^n f\|_{L^p(\mathbb{T})} + \|KP_\mathbb{T} U^n f\|_{L^p(\mathbb{T})} + \\ &\qquad \delta\|U^{-n}(I_{L^p(\mathbb{T})} - P_\mathbb{T})U^n f\|_{L^p(\mathbb{T})} \geq \delta\|f\|_{L^p(\mathbb{T})} \end{aligned} \tag{1.4.149}$$

for all $f \in L^p(\mathbb{T})$. Due to the fact of $U^n \xrightarrow[n\to\infty]{} 0$ weakly on $L^p(\mathbb{T})$ and being K a compact operator, we have

$$KP_\mathbb{T} U^n \xrightarrow[n\to\infty]{} 0 \text{ strongly on } L^p(\mathbb{T}). \tag{1.4.150}$$

Additionally, consider the dense subset $\mathcal{P}$ of $L^p(\mathbb{T})$ of all trigonometric polynomials

$$\sum_{k=-n}^{n} f_k t^k \quad (t \in \mathbb{T})\,. \tag{1.4.151}$$

Since in $\mathcal{P}$ the action of $P_{\mathbb{T}}$ consists of

$$P_{\mathbb{T}} : \sum_{k=-n}^{n} f_k t^k \mapsto \sum_{k=0}^{n} f_k t^k, \tag{1.4.152}$$

it is clear that $U^{-n} P_{\mathbb{T}} U^n f$ converges in $L^p(\mathbb{T})$ to f, for every $f \in \mathcal{P}$. This together with the uniformly boundedness of $U^{-n} P_{\mathbb{T}} U^n$ on $L^p(\mathbb{T})$ yields that

$$U^{-n} P_{\mathbb{T}} U^n \xrightarrow[n\to\infty]{} I_{L^p(\mathbb{T})} \text{ strongly on } L^p(\mathbb{T}). \tag{1.4.153}$$

A similar argument allows the conclusion that

$$U^{-n} P_{\mathbb{T}} U^{-n} \xrightarrow[n\to\infty]{} 0 \text{ strongly on } L^p(\mathbb{T}). \tag{1.4.154}$$

Using the identities $U^n U^{-n} = I_{L^p(\mathbb{T})}$ and $J_{\mathbb{T}} U^n = U^{-n} J_{\mathbb{T}}$, we can write

$$U^{-n} T_{B_0\phi}\, U^n = (U^{-n} P_{\mathbb{T}} U^n)(B_0\phi)\,(U^{-n} P_{\mathbb{T}} U^n), \tag{1.4.155}$$

$$U^{-n} H_{B_0\phi}\, U^n = (U^{-n} P_{\mathbb{T}} U^{-n})(B_0\phi)\, J_{\mathbb{T}}(U^{-n} P_{\mathbb{T}} U^n). \tag{1.4.156}$$

From (1.4.153) and (1.4.154), it follows that

$$U^{-n} T_{B_0\phi}\, U^n \xrightarrow[n\to\infty]{} (B_0\phi) I_{L^p(\mathbb{T})} \text{ strongly on } L^p(\mathbb{T}), \tag{1.4.157}$$

$$U^{-n} H_{B_0\phi}\, U^n \xrightarrow[n\to\infty]{} 0 \text{ strongly on } L^p(\mathbb{T}), \tag{1.4.158}$$

i.e.,

$$U^{-n}(T-H)_{B_0\phi}\, U^n \xrightarrow[n\to\infty]{} (B_0\phi) I_{L^p(\mathbb{T})} \text{ strongly on } L^p(\mathbb{T}). \tag{1.4.159}$$

Taking now the limit $n \to \infty$ in (1.4.149) and using (1.4.150), (1.4.153) and (1.4.159), we obtain

$$\|(B_0\phi) f\|_{L^p(\mathbb{T})} \geq \delta \|f\|_{L^p(\mathbb{T})} \tag{1.4.160}$$

for all $f \in L^p(\mathbb{T})$. Therefore $B_0\phi \in \mathcal{G}L^\infty(\mathbb{T})$ or, equivalently, $\phi \in \mathcal{G}L^\infty(\mathbb{R})$. Since $\|\phi\|_{L^\infty(\mathbb{R})} \leq \|\phi\|_{M^p(\mathbb{R})}$, it holds that $\phi \in \mathcal{G}M^p(\mathbb{R})$. The d-normal case follows from the n-normal case by passage to adjoint operators.

To prove that $\mathcal{R}(\phi) \subset sp_{ess}(W+H)_\phi$, let us consider $\lambda \in \mathcal{R}(\phi)$. Then $\psi = \phi - \lambda \notin \mathcal{GM}^p(\mathbb{R})$ and, from the first part of the theorem, $(W+H)_\psi$ is neither a n-normal or a d-normal operator. Thus $(W+H)_\psi$ is not a Fredholm operator, and so $\ell_0(W+H)_\psi$ is not also a Fredholm operator. Additionally, we have

$$\begin{aligned}\ell_0(W+H)_\psi &= \ell_0 W_\psi + \ell_0 H_\psi \\ &= \ell_0 W_\phi - \lambda \ell_0 r_+ \mathcal{F}^{-1}\mathcal{F} + \ell_0 H_\phi - \lambda \ell_0 r_+ \mathcal{F}^{-1}\mathcal{F} J \\ &= \ell_0(W+H)_\phi - \lambda I_{L^p_+(\mathbb{R})}. \end{aligned} \tag{1.4.161}$$

As $\ell_0(W+H)_\phi - \lambda I_{L^p_+(\mathbb{R})}$ is not a Fredholm operator, it follows that $\lambda \in sp_{ess}(W+H)_\phi$. The inclusion $sp_{ess}(W+H)_\phi \subset sp\ (W+H)_\phi$ is obvious from the definitions of these sets. □

Since a Fredholm operator is a n-normal and a d-normal operator, a trivial consequence of Theorem 1.4.1 is that $\phi \in \mathcal{G}L^\infty(\mathbb{R})$ if $(W+H)_\phi$ is a Fredholm operator. This weaker result can also be reached without appealing to Theorem 1.4.1 by using the equivalence relation presented in Lemma 1.3.11 to conclude that $(T-H)_{B_0\phi}$ is a Fredholm operator if $(W+H)_\phi$ is also a Fredholm operator. Then, applying [31, Proposition 2.1], it follows $B_0\phi \in \mathcal{G}L^\infty(\mathbb{T})$ (which is the same as $\phi \in \mathcal{G}L^\infty(\mathbb{R})$).

A direct consequence of Theorem 1.4.1 is the next result which shows us the reason why in the Fredholm and invertibility criteria presented in Chapters 3, 4, 5 and 6 we only consider operators with invertible symbols.

Corollary 1.4.2. *If $\phi \notin \mathcal{G}M^p(\mathbb{R})$, then $(W+H)_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$ is not a semi--Fredholm operator.*

Proof. The statement follows directly from Theorem 1.4.1. □

For the case of Wiener-Hopf minus Hankel operators, similar results hold true.

Theorem 1.4.3. *If $\phi \in M^p(\mathbb{R})$ and the operator*

$$(W-H)_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+) \tag{1.4.162}$$

is n-normal or d-normal, then $\phi \in \mathcal{G}M^p(\mathbb{R})$. *Consequently,*

$$\mathcal{R}(\phi) \subset sp_{ess}(W-H)_\phi \subset sp\,(W-H)_\phi. \tag{1.4.163}$$

Proof. Similar to the proof of Theorem 1.4.1. □

Corollary 1.4.4. *If* $\phi \notin \mathcal{G}M^p(\mathbb{R})$*, then* $(W-H)_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$ *is not a semi--Fredholm operator.*

Proof. Direct consequence of Theorem 1.4.3. □

Chapter 2

Fourier Symbols

This chapter is devoted to the algebras where the Fourier symbols of the Wiener-Hopf--Hankel operators under study belong to: the algebra of almost periodic functions, the algebra of semi-almost periodic functions, and the algebra of piecewise almost periodic functions. For each case, after a brief historical background, we present the definition of the algebra, some examples of elements of the algebra, as well as some properties that will be used in the next chapters. As we will see later, piecewise almost periodic functions are a generalization of semi-almost periodic functions and semi-almost periodic functions are a generalization of almost periodic functions. Moreover, for each piecewise almost periodic function or semi-almost periodic function, we identify two almost periodic functions, called their almost periodic representatives at minus and plus infinity. Taking this into account, we also present in the section devoted to almost periodic functions the definitions of some mean numbers. These mean numbers of the almost periodic functions or of the almost periodic representatives of semi-almost periodic or piecewise almost periodic functions will play a decisive role in the Fredholm and invertility criteria presented in Chapters 3, 4, 5 and 6 in order to decide if the operator is Fredholm, (properly) n-normal or (properly) d-normal, invertible, left-invertible or right-invertible.

2.1 Almost periodic functions

The theory of almost periodic functions was created by H. Bohr between 1923 and 1925. Since then, many contributions to the development of the theory of almost periodic functions were made, namely by V. V. Stepanov, H. Wyel, A. S. Besicovitch, S. Bochner, J. Von Neumann, C. Corduneanu, and others. The importance of almost periodic functions in problems of differential equations, stability theory and dynamical systems potentiated the development of the theory of this class of functions.

For defining almost periodic functions, H. Bohr used the concepts of *relative density* and *translation number* (cf. [11]). Concerning to relative density, a set $E \subset \mathbb{R}$ is said to be *relatively dense* if there exists a number $l > 0$ such that any interval of length l contains at least one number of E. To define a translation number, consider ϕ being a real or complex function defined on the real line. A number τ is called a *translation number* of ϕ, corresponding to $\varepsilon > 0$, if

$$|\phi(x+\tau) - \phi(x)| \leq \varepsilon \tag{2.1.1}$$

for all $x \in \mathbb{R}$. These two concepts provide the conditions to present the definition of almost periodic function: a continuous function ϕ defined on the real line is called *(uniformly) almost periodic* if for every $\varepsilon > 0$ there exists a relatively dense set of translation numbers of ϕ corresponding to ε. That is to say, ϕ is called an almost periodic function if for every $\varepsilon > 0$ there exists a number $l > 0$ such that any interval of length l contains at least one number τ for which

$$|\phi(x+\tau) - \phi(x)| \leq \varepsilon\,, \tag{2.1.2}$$

for all $x \in \mathbb{R}$. The class of almost periodic functions is usually denoted by AP.

To illustrate the notion of almost periodic function, Figures 2.1 and 2.2 exhibit the images of two particular almost periodic functions.

The theory of almost periodic functions was developed in analogy with the theory of periodic functions, since almost periodic functions are a generalization of periodic functions. In this way, some of the properties of periodic functions still hold true for almost periodic functions. Namely, every almost periodic function is bounded and uniformly continuous.

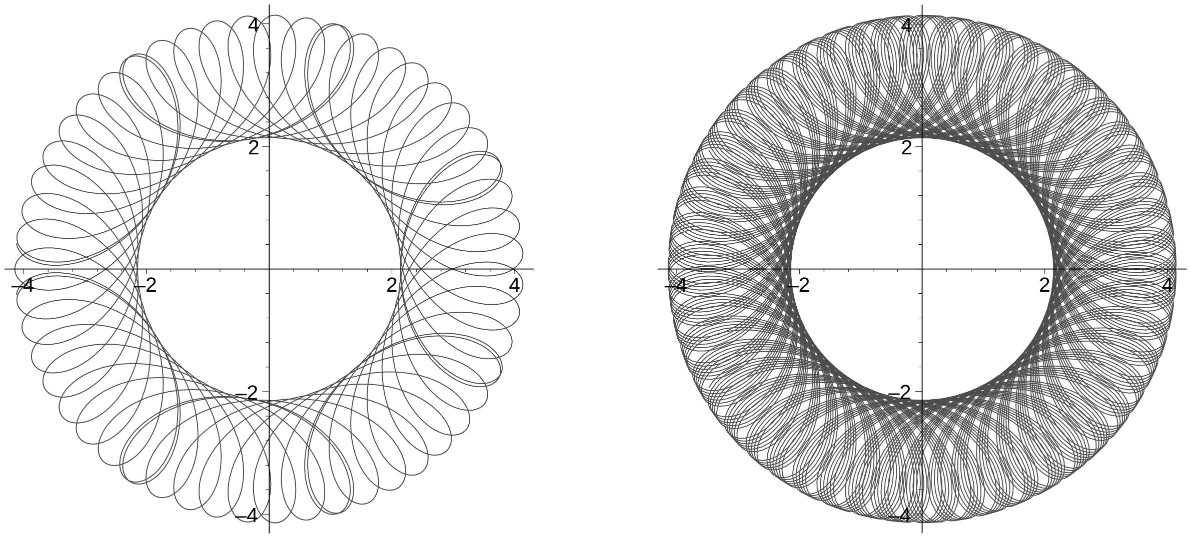


Figure 2.1: The image of $\phi(x) = -e^{i\pi x} - \pi e^{-\frac{ix}{2}}$ for x in $[-50, 50]$ and $[-200, 200]$, respectively.

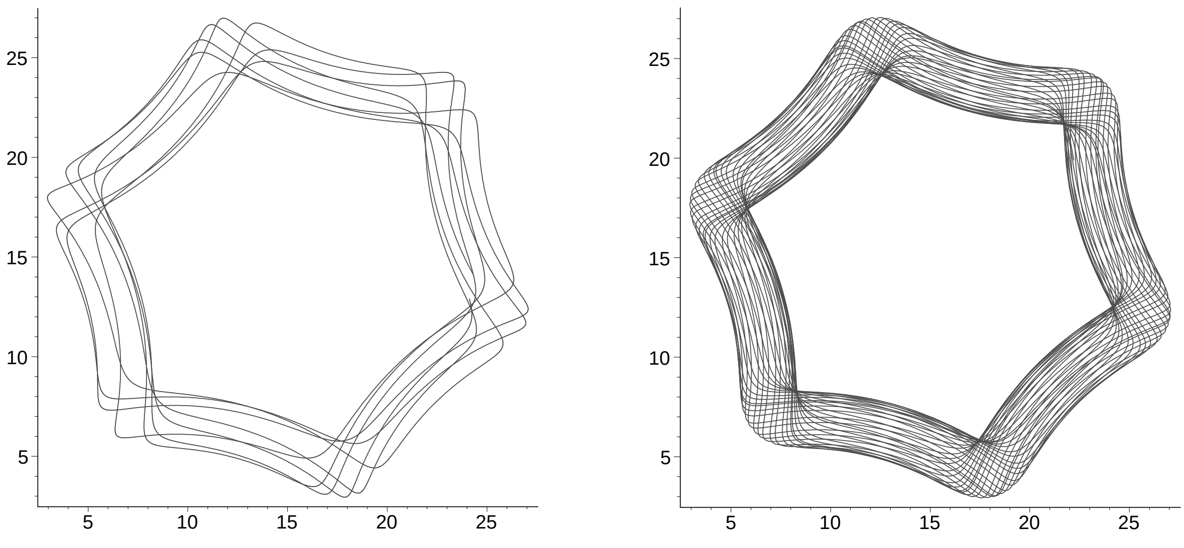


Figure 2.2: The image of $\phi(x) = 15 + 15i + 10e^{-ix} + (1+i)e^{i\sqrt{2}x} - ie^{i5x}$ for x in $[-25, 25]$ and $[-100, 100]$, respectively.

Therefore, due to the boundedness of almost periodic functions, we may define the norm

$$\|\phi\|_{AP} := \sup_{x\in\mathbb{R}} |f(x)|, \tag{2.1.3}$$

for $\phi \in AP$. It follows that AP is a Banach algebra when endowed with this norm and with respect to pointwise addition and multiplication.

Like the periodic functions, almost periodic functions can also be represented by Fourier series. To obtain such representation, we have to consider the mean value of an almost periodic function. The *Mean Value Theorem* (cf. [11, §50]) states that for every $\phi \in AP$ the limit

$$M(\phi) := \lim_{T\to\infty} \frac{1}{T} \int_0^T \phi(x)\, dx \tag{2.1.4}$$

exists and is finite. In this context, the number $M(\phi)$ is called the *Bohr mean value* of ϕ (or simply the *mean value* of ϕ). Figures 2.3 and 2.4 exhibit two examples of almost periodic functions and their corresponding mean values.

In [11, §52], H. Bohr proved the *Strengthened Mean Value Theorem*, that is, the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_a^{a+T} \phi(x)\, dx = M(\phi) \tag{2.1.5}$$

exists uniformly with respect to a ($a \in \mathbb{R}$). Due to this result, some authors present an equivalent definition of the Bohr mean value of ϕ, defining it as

$$M(\phi) = \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} \phi(x) dx, \tag{2.1.6}$$

where for an unbounded set $A \subset \mathbb{R}_+$, $\{I_\alpha\}_{\alpha\in A} = \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ is a family of intervals $I_\alpha \subset \mathbb{R}$ such that $|I_\alpha| = y_\alpha - x_\alpha \to \infty$, as $\alpha \to \infty$. In this case, we have that $M(\phi)$ is independent of the particular choice of the family $\{I_\alpha\}$ (see, e.g., [12, Proposition 2.22]).

For $\lambda \in \mathbb{R}$, let

$$e_\lambda(x) := e^{i\lambda x}, \quad x \in \mathbb{R}. \tag{2.1.7}$$

Considering $\phi \in AP$ and for each $\lambda \in \mathbb{R}$, it follows that the function $\phi\, e_{-\lambda}$, being the product of two almost periodic functions, is also an almost periodic function. Therefore, there exits the Bohr mean value of $\phi\, e_{-\lambda}$. In this way, for each $\phi \in AP$, define a corresponding

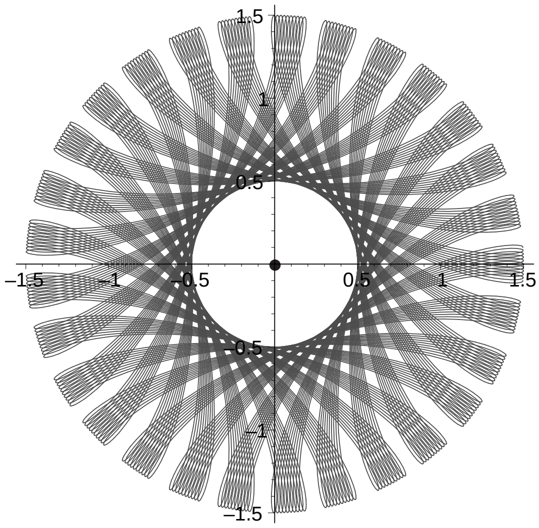


Figure 2.3: The image of $\phi(x) = \frac{1}{2}e^{-i\pi x} + e^{i\sqrt{2}x}$ for x in $[-200, 200]$ and its mean value (dot).

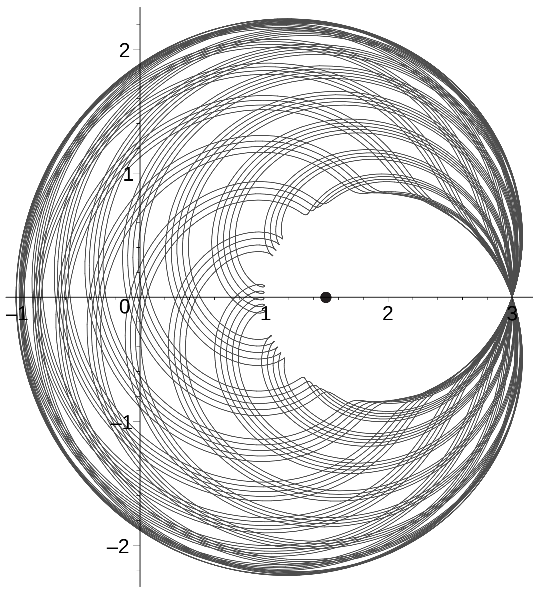


Figure 2.4: The image of $\phi(x) = 3e^{ix}\cos x + e^{i\pi x}\sin x$ for x in $[-100, 100]$ and its mean value (dot).

function φ given by $\varphi(\lambda) = M(\phi\, e_{-\lambda})$, $\lambda \in \mathbb{R}$. These corresponding functions φ have a property that will be very important in the representation of the almost periodic functions by a Fourier series. The property is the following: every function φ is zero for all values of λ with the exception of at most a countable set of numbers λ. Therefore, the set

$$\Omega(\phi) := \{\lambda \in \mathbb{R} : \varphi(\lambda) \neq 0\} \tag{2.1.8}$$

is at most countable, and is called the *Bohr-Fourier spectrum* of ϕ. For each $\lambda_j \in \Omega(\phi)$, consider $\phi_j = \varphi(\lambda_j)$. Clearly, since $\Omega(\phi)$ is an at most countable set, the set of all ϕ_j is also at most countable. Thus, the Fourier series associated with the function ϕ is given by

$$\sum_{\lambda_j \in \Omega(\phi)} \phi_j\, e_{\lambda_j}\,, \tag{2.1.9}$$

and so we may write

$$\phi(x) \sim \sum_{\lambda_j \in \Omega(\phi)} \phi_j\, e^{i\lambda_j x}\,. \tag{2.1.10}$$

From this representation of ϕ in terms of a Fourier series, it follows that the elements λ_j of the Bohr-Fourier spectrum $\Omega(\phi)$ are called the *Fourier exponents* of ϕ, and the corresponding mean values ϕ_j are called the *Fourier coefficients* of ϕ.

In [11, §84], we find the *Fundamental Theorem* of the theory of almost periodic functions. The Fundamental Theorem states that the class of almost periodic functions is identical with the closure of the class of all *trigonometric polynomials*:

$$p(x) = \sum_{n=1}^{N} \phi_n\, e^{i\lambda_n x}, \tag{2.1.11}$$

where $\phi_n \in \mathbb{C}$ and $\lambda_n \in \mathbb{R}$. To reach the Fundamental Theorem, H. Bohr proved that the limit of a uniformly convergent sequence of almost periodic functions is also an almost periodic function. Thus, being trigonometric polynomials almost periodic functions, it follows that the limit of a uniformly convergent sequence of trigonometric polynomials is an almost periodic function. This means that every function belonging to the closure of the class of all finite sums (2.1.11) is in the class of almost periodic functions. The converse also holds, i.e., every almost periodic function is the limit of a uniformly convergent sequence of trigonometric polynomials. This result is called the *Approximation Theorem* (cf. [11, §84]). More precisely, the Approximation Theorem asserts that given $\phi \in AP$, for each $\varepsilon > 0$, there exists a trigonometric polynomial p, whose exponents are the Fourier exponents of ϕ, and such that

$$|\phi(x) - p(x)| \leq \varepsilon \tag{2.1.12}$$

for all $x \in \mathbb{R}$.

From the Approximation Theorem, we obtain a characterization of almost periodic functions which is by some authors presented as the definition of almost periodic functions. See, e.g., [26, Chapter I] where almost periodic functions are those complex-valued functions defined on the real line which can be uniformly approximated by trigonometric polynomials, and [12, Definition 1.11] where the algebra of the almost periodic functions is defined as the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ ($\lambda \in \mathbb{R}$). From this last definition, we clearly see that AP is a C^*–subalgebra of $L^\infty(\mathbb{R})$.

H. Bohr also proved that the argument of an invertible almost periodic function is given by the sum of a linear function and an almost periodic function. More precisely, by *Bohr's Theorem* (see [10] and [50, pp. 49]), for each $\phi \in \mathcal{G}AP$ there exists a real number $\kappa(\phi)$ and a function $\psi \in AP$ such that

$$\phi(x) = e^{i\kappa(\phi)x} e^{\psi(x)}, \quad x \in \mathbb{R}. \tag{2.1.13}$$

Since $\kappa(\phi)$ is uniquely determined, $\kappa(\phi)$ is usually called the *mean motion* of ϕ. Considering $\phi \in \mathcal{G}AP$, the mean motion of ϕ can be obtained by

$$\kappa(\phi) = \lim_{T\to\infty} \frac{(\arg\phi)(T) - (\arg\phi)(0)}{T}, \tag{2.1.14}$$

where $\arg\phi$ is any continuous argument of ϕ. For an invertible bounded continuous function on $\mathbb{R}$, ϕ, a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ is said to be a *continuous argument* of ϕ, and is denoted by $\arg\phi$, if φ is such that $\phi = |\phi|e^{i\varphi}$. From the definition of continuous argument, it yields that there are countably many continuous arguments of ϕ since any two of continuous arguments of ϕ differ only by a multiple of 2π.

For every almost periodic function ϕ, we already associated one mean number: the Bohr mean value, $M(\phi)$. In the case where ϕ is also invertible, we also associated the mean motion, $\kappa(\phi)$. There is still one more mean number that we can associate to ϕ – the geometric mean value. For $\phi \in \mathcal{G}AP$, the number

$$\mathbf{d}(\phi) := e^{M(\psi)} \tag{2.1.15}$$

is called the *geometric mean value* of ϕ (where ψ is the same as in (2.1.13)). If $\kappa(\phi) = 0$, then we may represent $\mathbf{d}(\phi)$ as

$$\mathbf{d}(\phi) = e^{M(\log \phi)} \tag{2.1.16}$$

where $\log \phi$ is any function in AP for which $\phi = e^{\log \phi}$. The geometric mean value of an almost periodic function is an important characteristic since we may say that it plays the same role as the one-sided limits play for piecewise continuous functions.

As we have seen before, every almost periodic function may be represented by a series. Nevertheless, not every almost periodic function may be represented by an absolutely convergent series. Thus, we denote by APW the subclass of all functions $\varphi \in AP$ which can be written in the form of an absolutely convergent series:

$$\varphi(x) = \sum_j \varphi_j e^{i\lambda_j x} \;\; (x \in \mathbb{R}), \quad \lambda_j \in \mathbb{R}, \quad \varphi_j = M(\varphi\, e_{-\lambda_j}), \quad \sum_j |\varphi_j| < \infty\,. \tag{2.1.17}$$

APW is a Banach algebra with respect to pointwise addition and multiplication and with respect to the norm

$$\|\varphi\|_{APW} := \sum_j |\varphi_j|, \tag{2.1.18}$$

where φ_j are defined in (2.1.17). The algebras AP and APW are inverse closed in $L^\infty(\mathbb{R})$, i.e., if $\phi \in AP$ ($\phi \in APW$) is invertible in $L^\infty(\mathbb{R})$, then $\phi^{-1} \in AP$ ($\phi^{-1} \in APW$).

Next, we present the definitions of other special subsets of AP that will have a preponderant use in the factorizations of almost periodic functions proposed in Section 3.1. For that purpose, let AP^+ denote the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ with $\lambda \geq 0$, and AP^- the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ with $\lambda \le 0$, i.e.,

$$AP^+ := \mathrm{alg}_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \ge 0\}, \tag{2.1.19}$$

$$AP^- := \mathrm{alg}_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \le 0\}. \tag{2.1.20}$$

Clearly, we see that $AP^\pm$ are closed subalgebras of AP. Therefore, we have

$$AP^- = \{\phi \in AP : \Omega(\phi) \subset (-\infty, 0]\}, \tag{2.1.21}$$

$$AP^+ = \{\phi \in AP : \Omega(\phi) \subset [0, +\infty)\}. \tag{2.1.22}$$

Moreover, it also holds that $AP^\pm$ are closed subalgebras of $AP \cap H^\infty_\pm(\mathbb{R})$. Thus, $AP^- \cap AP^+ = \mathbb{C}$ since $H^\infty_-(\mathbb{R}) \cap H^\infty_+(\mathbb{R}) = \mathbb{C}$ (this conclusion follows also directly from (2.1.21) and (2.1.22)). Additionally, let APW^- (APW^+) be the set of all functions $\psi \in APW$ such that $\Omega(\psi) \subset (-\infty, 0]$ ($\Omega(\psi) \subset [0, +\infty)$). Obviously, $APW^\pm$ are closed subalgebras of APW, $APW^- \subset AP^-$ and $APW^+ \subset AP^+$.

To end up this section about almost periodic functions, we present a well-known characterization of invertible $AP^\pm$ functions.

Lemma 2.1.1. ([12, Lemma 3.4]) *Let $\phi \in AP^\pm$. $\phi \in \mathcal{G}AP^\pm$ if and only if there exists a $\psi \in AP^\pm$ such that $\phi = e^\psi$.*

For $APW^\pm$ functions, a similar characterization holds. That is, $\phi \in \mathcal{G}APW^\pm$ if and only if there exists a $\psi \in APW^\pm$ such that $\phi = e^\psi$. Combining now Lemma 2.1.1 with Bohr's Theorem (cf. (2.1.13)), it follows that $\phi \in \mathcal{G}AP^\pm$ if and only if $\kappa(\phi) = 0$.

For more information about almost periodic functions we refer to [9, 11, 26, 50].

2.2 Semi-almost periodic functions

We start this section by introducing some notation concerned with continuous functions. For that purpose, let $C(\dot{\mathbb{R}})$ denote the set of all (bounded) continuous (complex-valued) functions on $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, i.e., the set of all (bounded) continuous (complex-valued) functions on $\mathbb{R}$ for which both limits at $-\infty$ and at $+\infty$ exist and coincide. Further, let

$C_0(\dot{\mathbb{R}})$ denote the set of all functions in $C(\dot{\mathbb{R}})$ such that the limits at $-\infty$ and at $+\infty$ are equal to zero, and $C(\overline{\mathbb{R}})$ be the set of all (bounded) continuous (complex-valued) functions on $\mathbb{R}$ with a possible jump at ∞ (i.e., the limits at $-\infty$ and at $+\infty$ exist but could be distinct). Here and in what follows, $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

Before defining the algebra of semi-almost periodic functions, we will briefly recall how semi-almost periodic functions appear. Convolution operators with symbols in $AP+C_0(\dot{\mathbb{R}})$ were studied in first place by I. Gohberg and I. A. Feldman, and L. A. Coburn and R. G. Douglas. Then, due to an initial suggestion of I. Gohberg to D. Sarason, in view of the consideration at the same time of piecewise continuous and almost periodic symbols, D. Sarason [67] developed the semi-Fredholm theory for Toeplitz operators in the Hardy space H^2, with symbols in the algebra of semi-almost periodic elements. In fact, in this paper of D. Sarason, we find the origin of the semi-almost periodic functions. Later on, R. V. Duduchava and A. I. Saginašvili [29, 65] worked out the corresponding semi-Fredholm theory for Wiener-Hopf operators with semi-almost periodic Fourier symbols, and acting between L^p Lebesgue spaces ($1 < p < \infty$). All this was done upon conditions on the mean motions and on the geometric mean values of the almost periodic representatives of the Fourier symbols at minus and plus infinity.

The suggestion of I. Gohberg to D. Sarason lead D. Sarason to consider the algebra generated by almost periodic functions and continuous functions. Therefore, the C^*–algebra of the *semi-almost periodic functions* on $\mathbb{R}$ (usually denoted by SAP) is by definition the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains AP and $C(\overline{\mathbb{R}})$. I.e.,

$$SAP := \mathrm{alg}_{L^\infty(\mathbb{R})}(AP, C(\overline{\mathbb{R}})). \tag{2.2.23}$$

Consider now the product $\phi = \varphi\psi$, with $\varphi \in AP$ and $\psi \in C(\overline{\mathbb{R}})$. For $u \in C(\overline{\mathbb{R}})$ such that $u(-\infty) = 0$ and $u(+\infty) = 1$, it holds

$$\begin{aligned}\phi &= (1-u)\psi\varphi + u\psi\varphi\\ &= (1-u)(\psi(-\infty) + \psi_{0-})\varphi + u(\psi(+\infty) + \psi_{0+})\varphi\\ &= (1-u)\psi(-\infty)\varphi + u\psi(+\infty)\varphi + (1-u)\psi_{0-}\varphi + u\psi_{0+}\varphi,\end{aligned} \tag{2.2.24}$$

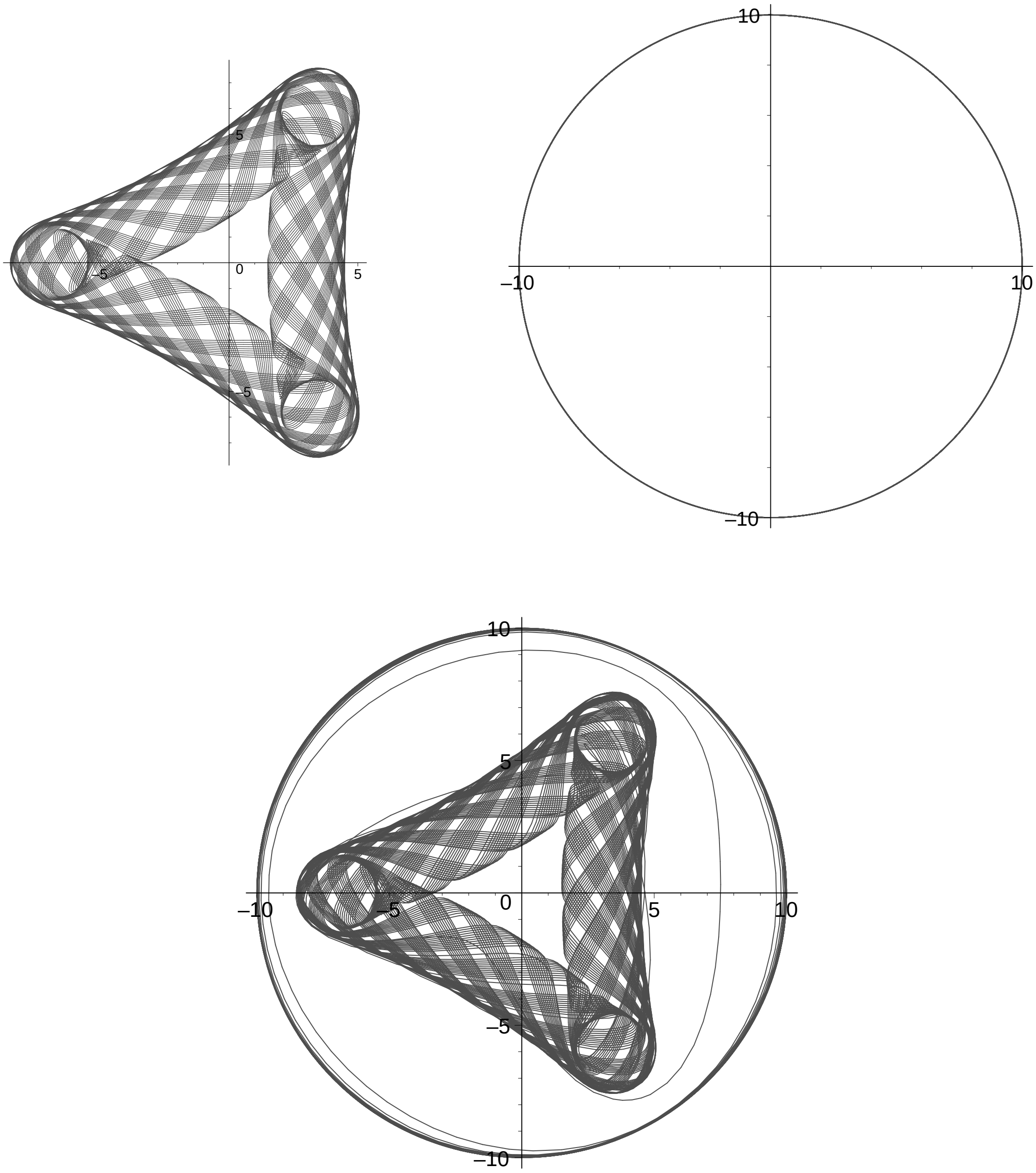


Figure 2.5: The range of $\phi_l(x) = -2e^{-2ix} + 5e^{ix} + \frac{3}{2}e^{i\pi x}$ for x in $[-200, 200]$ (upper left), of $\phi_r(x) = 10e^{ix}$ for x in $[-200, 200]$ (upper right), and of $\phi = (1-u)\phi_l + u\phi_r + \phi_0$ for x in $[-500, 500]$ (bottom), where $u(x) = \frac{1}{2} + \frac{1}{\pi}\arctan x$, and $\phi_0(x) = \frac{1}{x^2+1}$.

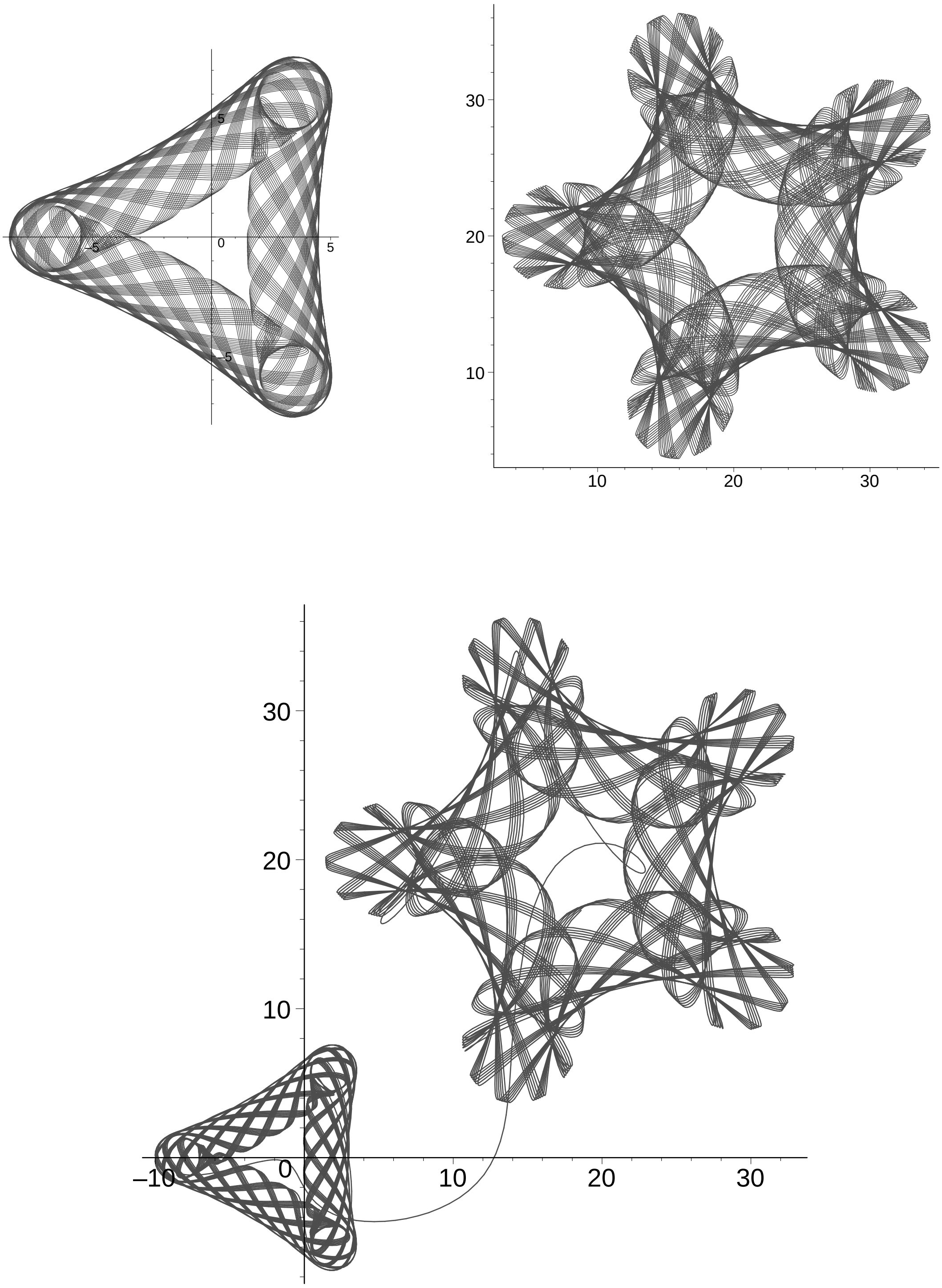


Figure 2.6: The range of $\phi_l(x) = -2e^{-2ix} + 5e^{ix} + \frac{3}{2}e^{i\pi x}$ for x in $[-200, 200]$ (upper left), of $\phi_r(x) = 20 + 20i - 4e^{-i4x} + 10ie^{ix} + 3e^{i\pi x}$ for x in $[-200, 200]$ (upper right), and of $\phi = (1-u)\phi_l + u\phi_r + \phi_0$ for x in $[-250, 250]$ (bottom), where $u(x) = \frac{1}{2}e^{x}$ if $x < 0$, $u(x) = 1 - \frac{1}{2}e^{-x}$ if $x \geq 0$, and $\phi_0(x) = \arctan(-|x|)$.

where $\psi_{0\pm} \in C(\overline{\mathbb{R}})$ are such that $\psi_{0\pm} = \psi - \psi(\pm\infty)$ (consequently $\psi_{0-}(-\infty) = \psi_{0+}(+\infty) = 0$). Being

$$\phi_l = \psi(-\infty)\varphi, \quad \phi_r = \psi(+\infty)\varphi, \quad \phi_0 = (1-u)\psi_{0-}\varphi + u\psi_{0+}\varphi, \tag{2.2.25}$$

it follows from (2.2.24) that

$$\phi = (1-u)\phi_l + u\phi_r + \phi_0, \tag{2.2.26}$$

where $\phi_l, \phi_r \in AP$ and $\phi_0 \in C_0(\dot{\mathbb{R}})$. From this, we obtain a well-known characterization of SAP (cf. [67]): for any $\phi \in SAP$ there exist $\phi_l, \phi_r \in AP$ and $\phi_0 \in C_0(\dot{\mathbb{R}})$ such that, for a fixed u in $C(\overline{\mathbb{R}})$ satisfying $u(-\infty) = 0$ and $u(+\infty) = 1$, it holds

$$\phi = (1-u)\phi_l + u\phi_r + \phi_0. \tag{2.2.27}$$

As we can see in (2.2.25), the functions ϕ_l, ϕ_r and ϕ_0 are uniquely determined by ϕ, and ϕ_l, ϕ_r are independent of the choice of u. The maps which perform the transformations

$$\phi \mapsto \phi_l, \quad \phi \mapsto \phi_r \tag{2.2.28}$$

are C^*–homomorphisms of SAP onto AP. In virtue of the uniqueness of ϕ_l and ϕ_r, ϕ_l and ϕ_r are usually called the *almost periodic representatives* of ϕ at $-\infty$ and $+\infty$, respectively. As an example, we present the images of two semi-almost periodic functions and also the images of the corresponding almost periodic representatives (see Figures 2.5 and 2.6).

Finally, we would like to mention that SAP is an inverse closed algebra in $L^\infty(\mathbb{R})$ (i.e., if $\phi \in SAP$ is invertible in $L^\infty(\mathbb{R})$, then $\phi^{-1} \in SAP$).

2.3 Piecewise almost periodic functions

Recalling the suggestion of I. Gohberg to D. Sarason, in view of the consideration of convolution operators with at the same time piecewise continuous and almost periodic symbols, the first results concerning to this were given by J. Xia [77], and A. Böttcher and B. Silbermann [14]. Although both papers were devoted to Wiener-Hopf operators with piecewise almost periodic functions, they have different goals. J. Xia studied the Fredholmness of certain subalgebras of Wiener-Hopf operators with piecewise almost periodic functions and presented a generalization of the index theory existent for Wiener-Hopf operators with almost periodic symbols in the context of von Neumann algebras for the case of Wiener-Hopf operators with piecewise almost periodic functions. On the other hand, A. Böttcher and B. Silbermann characterized the Fredholm property of all class of Wiener-Hopf operators with piecewise almost periodic functions but using arguments different from those of J. Xia.

In order to define the algebra of piecewise almost periodic functions, we need to introduce first some notions about piecewise continuous functions.

Consider the C^*-algebra of all *(bounded) piecewise continuous functions on* $\dot{\mathbb{R}}$ (usually denoted by PC or $PC(\dot{\mathbb{R}})$) as being the algebra of all functions $\psi \in L^\infty(\mathbb{R})$ for which the one-sided limits

$$\psi(x_0 - 0) := \lim_{x \to x_0 - 0} \psi(x)\,, \quad \psi(x_0 + 0) := \lim_{x \to x_0 + 0} \psi(x) \tag{2.3.29}$$

exist for each $x_0 \in \dot{\mathbb{R}}$. By convention, we take

$$\psi(\infty - 0) := \psi(+\infty) = \lim_{x \to +\infty} \psi(x)\,, \quad \psi(\infty + 0) := \psi(-\infty) = \lim_{x \to -\infty} \psi(x). \tag{2.3.30}$$

Since we identify functions which differ only on null measure sets, for $\psi \in PC$ we can assume that $\psi(x - 0) = \psi(x)$, for all $x \in \dot{\mathbb{R}}$. This means that it is enough to consider piecewise continuous functions which are always continuous from the left. Furthermore, let PC_0 be the sub-class of PC of all piecewise continuous functions for which both limits at $-\infty$ and at $+\infty$ are equal to zero.

Considering now $\phi = \psi\varphi$ such that $\psi \in PC$, $\varphi \in AP$, and $u \in C(\overline{\mathbb{R}})$ satisfying $u(-\infty) = 0$ and $u(+\infty) = 1$, it holds

$$\begin{aligned}
\phi &= (1-u)\psi\varphi + u\psi\varphi \\
&= (1-u)(\psi(-\infty) + \psi_{0-})\varphi + u(\psi(+\infty) + \psi_{0+})\varphi \\
&= (1-u)\psi(-\infty)\varphi + u\psi(+\infty)\varphi + (1-u)\psi_{0-}\varphi + u\psi_{0+}\varphi,
\end{aligned} \tag{2.3.31}$$

where $\psi_{0\pm} \in PC$ are such that $\psi_{0\pm} = \psi - \psi(\pm\infty)$. Put

$$\phi_l = \psi(-\infty)\varphi, \quad \phi_r = \psi(+\infty)\varphi, \quad \phi_0 = (1-u)\psi_{0-}\varphi + u\psi_{0+}\varphi. \tag{2.3.32}$$

From (2.3.31), we obtain

$$\phi = (1-u)\phi_l + u\phi_r + \phi_0, \tag{2.3.33}$$

where $\phi_l, \phi_r \in AP$ and $\phi_0 \in PC_0$ (please note that $\psi_{0-}(-\infty) = \psi_{0+}(+\infty) = 0$). After this, the definition of piecewise almost periodic function arises naturally. In this sense, the C^*-algebra of *piecewise almost periodic functions* on $\mathbb{R}$ (denoted by PAP) is by definition the collection of all functions of the form

$$\phi := (1-u)\phi_l + u\phi_r + \phi_0, \tag{2.3.34}$$

where $\phi_l, \phi_r \in AP$, $\phi_0 \in PC_0$ and $u \in C(\overline{\mathbb{R}})$ such that $u(-\infty) = 0$ and $u(+\infty) = 1$. From the definition of piecewise almost periodic functions, it can be shown that PAP is the algebra generated by AP and PC (cf. (2.3.31)), i.e.,

$$PAP = \mathrm{alg}_{L^\infty(\mathbb{R})}(AP, PC). \tag{2.3.35}$$

From (2.3.32), we see that the functions ϕ_l, ϕ_r and ϕ_0 are uniquely determined by ϕ, and ϕ_l, ϕ_r are independent of the choice of u. Similarly to the semi-almost periodic functions case, the maps which perform the transformations

$$\phi \mapsto \phi_l, \quad \phi \mapsto \phi_r \tag{2.3.36}$$

are C^*–homomorphisms of PAP onto AP. Due to all this, the functions ϕ_l and ϕ_r are called the *almost periodic representatives of* ϕ at $-\infty$ and $+\infty$, respectively.

Consider $\phi \in \mathcal{G}PAP$. Taking into account the representation (2.3.34) of ϕ, we define by

$$\kappa_l(\phi) := \kappa(\phi_l), \quad \kappa_r(\phi) := \kappa(\phi_r), \quad \mathbf{d}_l(\phi) := \mathbf{d}(\phi_l), \quad \mathbf{d}_r(\phi) := \mathbf{d}(\phi_r) \tag{2.3.37}$$

the *left and right mean motions, and the left and right geometric mean values* of ϕ, respectively.

For the case of invertible elements in PAP, it is possible to give an alternative representation to (2.3.34) in terms of a product between an invertible semi-almost periodic function and an invertible piecewise continuous function.

Proposition 2.3.1. ([12, Proposition 3.15]) *If $\phi \in \mathcal{G}PAP$, then there exist $\varphi \in \mathcal{G}SAP$ and $\psi \in \mathcal{G}PC$ satisfying $\psi(-\infty) = \psi(+\infty) = 1$ such that*

$$\phi = \varphi\,\psi. \tag{2.3.38}$$

Proof. From the definition of PAP, we have

$$\phi = (1-u)\phi_l + u\phi_r + \phi_0, \tag{2.3.39}$$

where $\phi_l, \phi_r \in AP$, $\phi_0 \in PC_0$ and $u \in C(\overline{\mathbb{R}})$ satisfying $u(-\infty) = 0$ and $u(+\infty) = 1$. Considering $\alpha = (1-u)\phi_l + u\phi_r$, there exists an $M > 0$ such that $|\alpha(x)|$ is bounded away from zero for $|x| > M$. Therefore, it is possible to find a function $\alpha_0 \in C_0(\dot{\mathbb{R}})$ such that $\varphi = \alpha + \alpha_0 \in \mathcal{G}SAP$. Thus, we have

$$\phi = \varphi + \phi_0 - \alpha_0 = \varphi\big(1 + \varphi^{-1}(\phi_0 - \alpha_0)\big) = \varphi\psi, \tag{2.3.40}$$

where $\psi = 1 + \varphi^{-1}(\phi_0 - \alpha_0)$. Since the limits at infinity of ϕ_0 and α_0 are equal to zero, it follows that $\psi(-\infty) = \psi(+\infty) = 1$. This means that $\psi \notin SAP$. In fact, due to the discontinuities of ϕ_0, it holds that $\psi \in PC$. From (2.3.40), we obtain $\psi = \varphi^{-1}\phi$ and so we conclude that ψ is also invertible in $L^\infty(\mathbb{R})$ in virtue of the invertibility of ϕ and φ. Finally, from the fact of PC being a C^*- subalgebra of $L^\infty(\mathbb{R})$ that contains the identity element of $L^\infty(\mathbb{R})$, it holds that PC is an inverse closed subalgebra in $L^\infty(\mathbb{R})$, i.e. $\psi \in PC \cap \mathcal{G}L^\infty(\mathbb{R})$ implies $\psi \in \mathcal{G}PC$. □

In order to exemplify the above result, we present an example of an invertible piecewise almost periodic function for which we exhibit the semi-almost periodic and the piecewise continuous functions that appear in (2.3.38).

Example 2.3.2. Consider the function ϕ (see Figure 2.7) given by

$$\phi(x) = (1-u(x))\,\pi\, e^{2ix} + u(x)\, e^{-ix} + \phi_0(x), \tag{2.3.41}$$

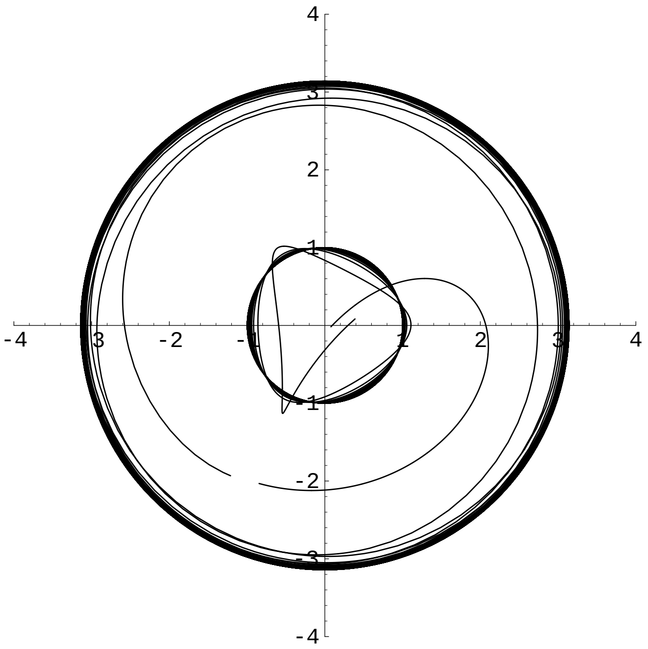


Figure 2.7: The range of $\phi(x)$ defined in (2.3.41) for x in $[-500, 500]$.

where

$$u(x) = \frac{1}{2} + \frac{1}{\pi}\arctan x \quad \text{ and } \quad \phi_0(x) = \begin{cases} -e^{x^3} & \text{if } x \leq -1 \\ -i\dfrac{x^2}{10} & \text{if } -1 < x \leq 1 \\ e^{-x}\sin x & \text{if } x > 1 \end{cases} . \tag{2.3.42}$$

Since $\phi \in \mathcal{G}PAP$ (cf. Figure 2.7), we may look for a representation of ϕ as in (2.3.38). For instance, considering φ and ψ given by

$$\varphi(x) = (1 - u(x))\,\pi\, e^{2ix} + u(x)\, e^{-ix} - ie^{-x^2} \tag{2.3.43}$$

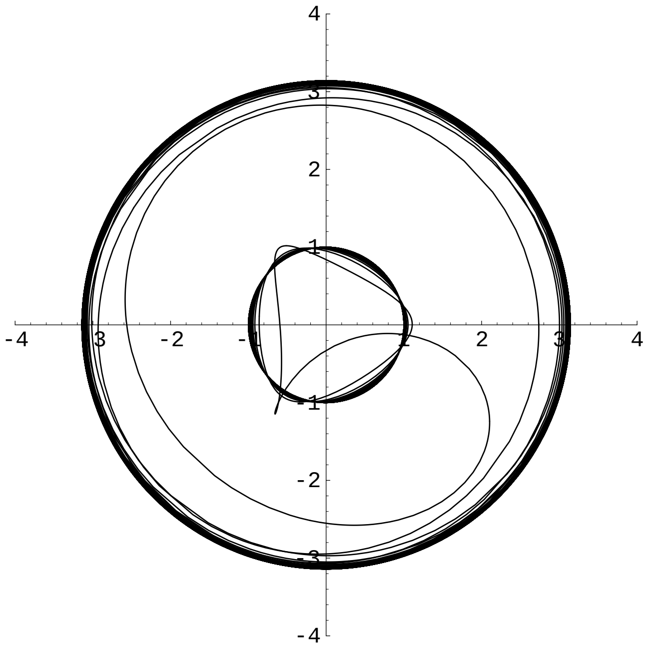


Figure 2.8: The range of $\varphi(x)$ presented in (2.3.43) for x in $[-500, 500]$.

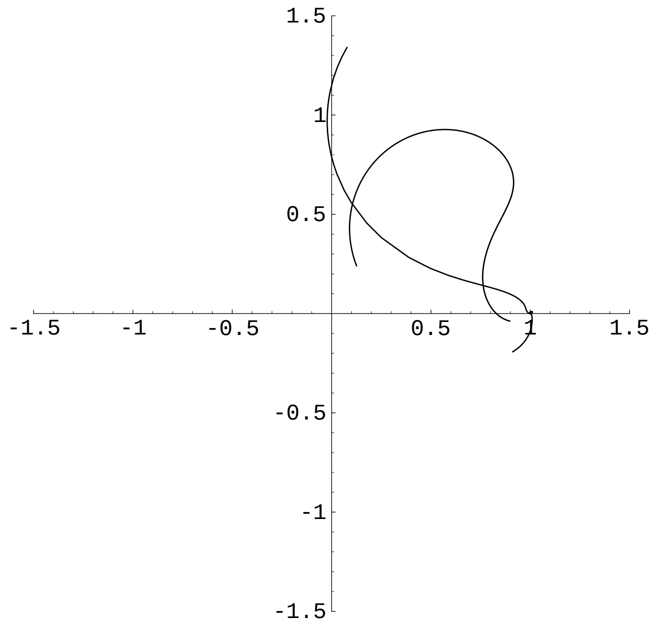


Figure 2.9: The range of $\psi(x)$ presented in (2.3.44) for x in $[-500, 500]$.

and

$$\psi(x) = \Big(1 + \varphi^{-1}\big(\phi_0 + ie^{-x^2}\big)\Big)(x) = \begin{cases} 1 + \dfrac{ie^{-x^2} - e^{x^3}}{\varphi(x)} & \text{if } x \leq -1 \\ 1 + \dfrac{ie^{-x^2} - i\dfrac{x^2}{10}}{\varphi(x)} & \text{if } -1 < x \leq 1, \\ 1 + \dfrac{ie^{-x^2} + e^{-x}\sin x}{\varphi(x)} & \text{if } x > 1 \end{cases} \tag{2.3.44}$$

respectively, we get

$$\phi = \varphi\,\psi. \tag{2.3.45}$$

Figures 2.8 and 2.9 confirm that $\varphi \in SAP$ and $\psi \in PC$ are invertible elements. Moreover, by (2.3.44), we get $\psi(-\infty) = \psi(+\infty) = 1$.

Clearly, we have that functions φ and ψ defined in (2.3.43) and (2.3.44), respectively, are not unique in the representation of ϕ in (2.3.38). In fact, if we had chosen another function belonging to $C_0(\dot{\mathbb{R}})$ in (2.3.43) such that $\varphi \in \mathcal{G}SAP$, we would obtain corresponding different semi-almost periodic and piecewise continuous functions φ and ψ.

Chapter 3

Wiener-Hopf plus Hankel Operators with Almost Periodic Symbols

For Wiener-Hopf operators with Fourier symbols in the algebra of almost periodic functions, acting between L^2 Lebesgue spaces, there is a famous invertibility and semi--Fredholm criterion based on the sign of the mean motion of the Fourier symbol of the operator. This criterion, which was obtained independently by L. A. Coburn and R. G. Douglas [25], and I. Gohberg and I. A. Feldman [37], states the following:

Theorem 3.0.3. ([12, Theorem 2.28]) *Let $\phi \in AP$ and suppose that ϕ is not identically zero.*

(a) *If $\phi \notin \mathcal{G}AP$, then W_ϕ is not normally solvable.*

(b) *If $\phi \in \mathcal{G}AP$ and $k(\phi) < 0$, then W_ϕ is properly d-normal and right-invertible.*

(c) *If $\phi \in \mathcal{G}AP$ and $k(\phi) > 0$, then W_ϕ is properly n-normal and left-invertible.*

(d) *If $\phi \in \mathcal{G}AP$ and $k(\phi) = 0$, then W_ϕ is invertible.*

Motivated by this result, the main purpose of this chapter is to establish an analogue invertibility and semi-Fredholm criterion for Wiener-Hopf plus Hankel operators with almost periodic Fourier symbols, acting between L^2 Lebesgue spaces.

In a first place, we begin by studying the invertibility and semi-Fredholm properties of Wiener-Hopf plus Hankel operators with Fourier symbols in the subalgebra of almost periodic functions APW. To reach such a criterion, a factorization theory is proposed for the Wiener-Hopf plus Hankel operators. We introduce a new factorization concept for the APW functions – the so-called APW asymmetric factorization – such that the properties of the factors will allow corresponding operator factorizations. As we will see in Theorem 3.1.3, every invertible APW function admits an APW asymmetric factorization, where the index of the middle factor of the factorization corresponds to the mean motion of the function. From here, conditions for left, right, both-sided invertibility, properly d-normal or properly n-normal of the Wiener-Hopf plus Hankel operators are obtained upon the mean motion of the Fourier symbol of the operators (cf. Theorem 3.2.1). Under such conditions, we are able to provide a formula for the one-sided and two-sided inverses of the Wiener-Hopf plus Hankel operators by using the factors of the APW asymmetric factorization. This result is stated in Theorem 3.2.2, and exhibits the importance of having convenient factorizations for the Fourier symbols of the corresponding operators.

Still concerning to Wiener-Hopf plus Hankel operators with APW Fourier symbols, we also present a result on the invertibility dependencies between Wiener-Hopf and Wiener--Hopf plus Hankel operators with the same APW Fourier symbol (see Theorem 3.2.3). At a first glance, this may appear to be a very surprising result. Noticing however that in this case we are dealing with a particular kind of factorization, it is more natural to hope to achieve the invertibility of Wiener-Hopf plus Hankel operators from the invertibility of Wiener-Hopf operators. In fact, to obtain this result, we need to introduce a new APW factorization – the APW antisymmetric factorization – as an analogue of factorizations used in [6, 30] for a different situation. This factorization allows to relate the APW asymmetric factorization with the well-known right APW factorization, being this the key result to reach the invertibility dependency between Wiener-Hopf and Wiener-Hopf plus Hankel operators.

After this study being complete, we consider the more general case of Wiener-Hopf plus Hankel operators with almost periodic Fourier symbols. Although this case runs analogously to the case of Wiener-Hopf plus Hankel operators with APW Fourier symbols in the greater part of the results, there are some differences. The reason is that in the AP case we cannot guarantee that every invertible AP function admits an AP asymmetric factorization, contrarily to what happens in the APW case, where every invertible APW function admits an APW asymmetric factorization (with the index of the middle factor of the factorization corresponding to the mean motion of the factorized function). Therefore, in the invertibility and semi-Fredholm criterion and, consequently, in order to provide a formula for the one-sided and two-sided inverses of the operators, we have to assume that the Fourier symbol of the Wiener-Hopf plus Hankel operator admits an AP asymmetric factorization (cf. Theorems 3.3.1 and 3.3.2). Due to this assumption and since we also cannot identify the index of the middle factor of the AP factorization with the mean motion of the function, Theorems 3.3.1 and 3.3.2 are obtained upon the index of the middle factor of the AP factorization instead of the mean motion of the Fourier symbol of the operator. Finally, like in the APW case, we can also establish invertibility dependencies between Wiener-Hopf and Wiener-Hopf plus Hankel operators with the same AP Fourier symbol. However, in this case, we have to consider an additional assumption related with the right AP factorization of the Fourier symbol, as we can see in Theorem 3.3.3.

The chapter is organized as follows. In the first section, we introduce the definitions of APW asymmetric factorization, APW antisymmetric, AP asymmetric factorization and AP antisymmetric factorization, as well as some results concerning to these factorizations that will be used in the obtainment of the results presented in the subsequent sections. The second section is devoted to the invertibility and semi-Fredholm criterion for Wiener-Hopf plus Hankel operators with APW Fourier symbols, while in the third section we present the invertibility and semi-Fredholm criterion for Wiener-Hopf plus Hankel operators with AP Fourier symbols. Seeing that the AP case follows as a generalization of the APW case, some results of the AP case have similar proofs of the analogous APW results. In those cases, we decided to omit the proofs.

3.1 AP factorizations

Factorization theory is a well-known technique to study the Fredholm property of several classes of operators, namely, Wiener-Hopf, Toeplitz and singular integral operators related to different curves and weighted spaces. Due to this, there exist several types of factorizations. As far as we know, the first factorization that appeared (due to this goal) was the Wiener-Hopf factorization. Although the most popular definition of this type of factorization is on the context of the unit circle, since we are dealing with operators acting on the real line, let us recall the analogue definition for the real line case.

A function $\phi \in \mathcal{G}L^\infty(\mathbb{R})$ is said to admit a *right Wiener-Hopf factorization* if it can be represented in the form

$$\phi(x) = \phi_-(x) \left(\frac{x-i}{x+i}\right)^\kappa \phi_+(x) \tag{3.1.1}$$

for almost all $x \in \mathbb{R}$, where κ is an integer and the factors ϕ_- and ϕ_+ satisfy the following conditions:

(i) $\dfrac{\phi_-(x)}{x-i} \in H^2_-(\mathbb{R})$, $\dfrac{\phi_-^{-1}(x)}{x-i} \in H^2_-(\mathbb{R})$,

(ii) $\dfrac{\phi_+(x)}{x+i} \in H^2_+(\mathbb{R})$, $\dfrac{\phi_+^{-1}(x)}{x+i} \in H^2_+(\mathbb{R})$,

(iii) the operator $\phi_- S \phi_-^{-1} I$ is bounded on $L^2(\mathbb{R})$.

Due to the uniqueness of κ, it is called the *right index of* ϕ. Moreover, a right Wiener-Hopf factorization with $\kappa = 0$ is referred to as a *canonical right Wiener-Hopf factorization.*

The name of the factorization presented above - the Wiener-Hopf factorization - is due to the factorization method proposed in 1931 by N. Wiener and E. Hopf for solving Wiener--Hopf integral equations. In spite of this, the Wiener-Hopf factorization has its origins in the work of F. D. Gakhov [33], who solved the Riemann-Hilbert problem with Hölder continuous coefficients on smooth Jordan curves, using the method of the Wiener-Hopf factorization.

The importance of this kind of factorization is given by a result of I. B. Simonenko ([70, 72]) that states the following: for $\phi \in L^\infty(\mathbb{R})$, the Wiener-Hopf operator W_ϕ is Fredholm on $L^2(\mathbb{R})$ if and only if $\phi \in \mathcal{G}L^\infty(\mathbb{R})$ and ϕ admits a right Wiener-Hopf factorization. In this case, the defect numbers are given by

$$n(W_\phi) = \max\{0, -\kappa\},\ d(W_\phi) = \max\{0, \kappa\}, \tag{3.1.2}$$

and hence

$$\operatorname{Ind} W_\phi = -\kappa, \tag{3.1.3}$$

where κ is the right index of ϕ. In particular, the Wiener-Hopf operator W_ϕ is invertible if and only if ϕ admits a canonical right Wiener-Hopf factorization.

Considering now almost periodic functions, there is a factorization for almost periodic functions that follows the factorization presented in Bohr's Theorem (cf. (2.1.13)) and also the Wiener-Hopf factorization. This factorization is the *almost periodic factorization* and it was introduced by Yu. I. Karlovich and I. M. Spitkovsky in [45] for matrix almost periodic functions.

Recall that a function $\phi \in \mathcal{G}AP$ is said to admit a *right AP factorization* (see e.g. [12, § 6.3]) if

$$\phi = \varphi_- e_\lambda \varphi_+ \tag{3.1.4}$$

where $\varphi_- \in \mathcal{G}AP^-$, $\varphi_+ \in \mathcal{G}AP^+$, and $\lambda \in \mathbb{R}$. In addition, if $\lambda = 0$ this factorization is called a *canonical right AP factorization*. In [12, § 6.3], we also find an analogue of the right AP factorization for APW functions, that is, the right APW factorization. Thus, a function $\phi \in \mathcal{G}APW$ is said to admit a *right APW factorization* if

$$\phi = \varphi_- e_\lambda \varphi_+ \tag{3.1.5}$$

where $\varphi_- \in \mathcal{G}APW^-$, $\varphi_+ \in \mathcal{G}APW^+$, and $\lambda \in \mathbb{R}$. In the case where $\lambda = 0$, the factorization is called a *canonical right APW factorization*.

In both cases, if ϕ admits a right AP or a right APW factorization, it follows that W_ϕ is equivalent to W_{e_λ}, since, by Theorem 3.0.3, W_{ϕ_-} and W_{ϕ_+} are invertible operators (see also the comment that follows Lemma 2.1.1). Therefore, from Theorem 3.0.3, it holds that if $\phi \in \mathcal{G}AP$ admits a right AP factorization or if $\phi \in \mathcal{G}APW$ admits a right APW

factorization and (a) $\lambda > 0$, then W_ϕ is properly n-normal and left-invertible; (b) $\lambda < 0$, then W_ϕ is properly d-normal and right-invertible; and (c) $\lambda = 0$, then W_ϕ is invertible. From here, we obtain a similar result of the Simonenko's result mentioned before. That is, for $\phi \in AP$ (resp. $\phi \in APW$), the Wiener-Hopf operator W_ϕ is invertible if and only if $\phi \in \mathcal{G}AP$ (resp. $\phi \in \mathcal{G}APW$) and ϕ admits a canonical right AP factorization (resp. canonical right APW factorization).

We would like to mentioned that by symmetry it is possible to define the "left" analogue of the right factorizations presented above, i.e., the left Wiener-Hopf factorization and the left AP (APW) factorization (cf. e.g. [12, § 6.2 and § 6.3]).

Having present all the right factorizations, let us see if we can obtain a factorization for the almost periodic functions in such a way that it will be possible to factorize the Wiener-Hopf-Hankel operators in order to obtain a characterization for the invertibility and the semi-Fredholm property of these operators.

In [6], E. L. Basor and T. Ehrhardt introduced the notion of asymmetric factorization of a function $\phi \in \mathcal{G}L^\infty(\mathbb{T})$ in the space $L^p(\mathbb{T})$ (with $1 < p < \infty$). In order to present the definition of the asymmetric factorization, we will first introduce some notation. Let $L^p_{\mathrm{even}}(\mathbb{T})$ denote the set of all functions $\phi \in L^p(\mathbb{T})$ which are even, i.e., the set of all functions $\phi \in L^p(\mathbb{T})$ such that $\phi(t) = \phi(t^{-1})$. Considering $P_{J_\mathbb{T}} := \frac{I+J_\mathbb{T}}{2}$, let $L^p_{J_\mathbb{T}}(\mathbb{T}) := \operatorname{Im} P_{J_\mathbb{T}}|_{L^p(\mathbb{T})}$. Additionally, recall that $\mathcal{P}$ denotes the linear space of all trigonometric polynomials (cf. (1.4.151)). After all this, we said that a function $\phi \in \mathcal{G}L^\infty(\mathbb{T})$ admits an *asymmetric factorization* in $L^p(\mathbb{T})$ if it can be represented in the form

$$\phi(t) = \phi_-(t) t^\kappa \phi_0(t), \quad t \in \mathbb{T}, \tag{3.1.6}$$

where $\kappa \in \mathbb{Z}$ and the factors ϕ_- and ϕ_0 satisfy the following conditions:

(i) $(1 + t^{-1})\phi_- \in H^p_-(\mathbb{T})$, $(1 - t^{-1})\phi_-^{-1} \in H^q_-(\mathbb{T})$,

(ii) $|1 - t|\phi_0 \in L^q_{\mathrm{even}}(\mathbb{T})$, $|1 + t|\phi_0^{-1} \in L^p_{\mathrm{even}}(\mathbb{T})$,

(iii) the linear operator $V := \phi_0^{-1}(I + J_\mathbb{T})P_\mathbb{T}\phi_-^{-1}I$ acting from X_1 into X_2 extends to a linear bounded operator V_e acting from $L^p(\mathbb{T})$ into $L^p_{J_\mathbb{T}}(\mathbb{T})$, where $\dfrac{1}{p}+\dfrac{1}{q}=1$, and

$$X_1=\{(1-t^{-1})\varphi(t):\varphi\in\mathcal{P}\},\quad X_2=\{(1+t^{-1})\phi_0^{-1}(t)\varphi(t):\varphi\in\mathcal{P},\varphi(t)=\varphi(t^{-1})\}. \tag{3.1.7}$$

With the help of this kind of factorization, E. L. Basor and T. Ehrhardt established a necessary and sufficient condition for the invertibility of the Toeplitz plus Hankel operator in terms of the asymmetric factorization of the Fourier symbol of the operator. The obtained condition states the following: considering $\phi\in\mathcal{G}L^\infty(\mathbb{T})$, the Toeplitz plus Hankel operator $(T{+}H)_\phi$ is invertible on $H^p_+(\mathbb{T})$ if and only if ϕ admits an asymmetric factorization in $L^p(\mathbb{T})$ with index $\kappa=0$. Similar to the role of the right Wiener-Hopf factorization in Simonenko's result, the importance of the asymmetric factorization is exhibited in this necessary and sufficient condition.

Motivated by these recent results on Toeplitz plus Hankel operators and also by the recent work on convolution type operators with symmetry [20, 22, 23], and having in mind the role of the APW factorization in the theory of Wiener-Hopf operators with APW symbols, we introduce a new type of APW factorization - the APW asymmetric factorization.

Definition 3.1.1. *We will say that a function $\phi\in\mathcal{G}APW$ admits an APW asymmetric factorization if it can be represented in the form*

$$\phi=\phi_-\,e_\lambda\,\phi_e \tag{3.1.8}$$

where $\phi_-\in\mathcal{G}APW^-$, $\phi_e\in\mathcal{G}L^\infty(\mathbb{R})$, $\widetilde{\phi_e}=\phi_e$ and $\lambda\in\mathbb{R}$ (recall that $e_\lambda(x)=e^{i\lambda x}$, $x\in\mathbb{R}$). The particular case of an APW asymmetric factorization with $\lambda=0$ will be referred to as a canonical APW asymmetric factorization.

Such as the asymmetric factorization presented by E. L. Basor and T. Ehrhardt, the APW asymmetric factorization here introduced is also unique up to a constant.

Proposition 3.1.2. *Let $\phi\in\mathcal{G}APW$. Suppose that ϕ admits two APW asymmetric* *factorizations:*

$$\phi = \phi_-^{(1)} \, e_{\lambda_1} \, \phi_e^{(1)} \,, \tag{3.1.9}$$

$$\phi = \phi_-^{(2)} \, e_{\lambda_2} \, \phi_e^{(2)} \,. \tag{3.1.10}$$

Then $\lambda_1 = \lambda_2$, $\phi_-^{(1)} = \gamma\phi_-^{(2)}$ *and* $\phi_e^{(1)} = \gamma^{-1}\phi_e^{(2)}$, $\gamma \in \mathbb{C} \setminus \{0\}$.

Proof. The equality $\phi_-^{(1)} \, e_{\lambda_1} \, \phi_e^{(1)} = \phi_-^{(2)} \, e_{\lambda_2} \, \phi_e^{(2)}$, implies that

$$(\phi_-^{(2)})^{-1} \, \phi_-^{(1)} \, e_{\lambda_1} \; = e_{\lambda_2} \, \phi_e^{(2)} \, (\phi_e^{(1)})^{-1}. \tag{3.1.11}$$

Assume, without loss of generality, that $\lambda_1 \leq \lambda_2$. Then $\lambda = \lambda_1 - \lambda_2 \leq 0$. From (3.1.11) it follows that

$$(\phi_-^{(2)})^{-1} \, \phi_-^{(1)} \, e_{\lambda} \; = \phi_e^{(2)} \, (\phi_e^{(1)})^{-1}. \tag{3.1.12}$$

Since the right-hand side of (3.1.12) is an even function, $(\phi_-^{(2)})^{-1} \, \phi_-^{(1)} \, e_\lambda$ is also an even function. Put

$$\varphi = (\phi_-^{(2)})^{-1} \, \phi_-^{(1)}. \tag{3.1.13}$$

Thus

$$\varphi(x) \, e_\lambda(x) = \widetilde{\varphi}(x) \, \widetilde{e_\lambda}(x), \tag{3.1.14}$$

i.e.,

$$\varphi(x) \, e_\lambda(x) = \widetilde{\varphi}(x) \, e_{-\lambda}(x), \tag{3.1.15}$$

or equivalently

$$\varphi(x) \, e_{2\lambda}(x) = \widetilde{\varphi}(x). \tag{3.1.16}$$

On one hand, since $\varphi \in \mathcal{G}APW^-$, we may apply the well-known characterization of $\mathcal{G}APW^-$ which assures the existence of a function $\psi \in APW^-$ such that $\varphi = e^\psi$ (see the comment that follows Lemma 2.1.1). On the other hand, since $\varphi \in \mathcal{G}APW^-$, we have $\widetilde{\varphi} \in \mathcal{G}APW^+$, and therefore, by a corresponding characterization of $\mathcal{G}APW^+$ (cf. Lemma 2.1.1), there exists a function $\eta \in APW^+$ such that $\widetilde{\varphi} = e^\eta$. From (3.1.16), it follows that

$$e^{\psi(x)+i2\lambda x} = e^{\eta(x)}, \tag{3.1.17}$$

which implies that $\lambda = 0$ and $\psi \in APW^- \cap APW^+$, i.e., $\lambda_1 = \lambda_2$ and ψ is a constant function. Finally, from (3.1.13), we get $\phi_-^{(1)} = \gamma\phi_-^{(2)}$ with $\gamma \in \mathbb{C}\backslash\{0\}$, and by (3.1.12), we obtain $\phi_e^{(1)} = \gamma^{-1}\phi_e^{(2)}$. □

The next result shows that every invertible function in APW possesses an APW asymmetric factorization, and reveals through this the first indicator of the importance of such a factorization.

Theorem 3.1.3. *If $\phi \in \mathcal{G}APW$, then ϕ admits an APW asymmetric factorization of the form*

$$\phi = \phi_- \; e_{\kappa(\phi)} \; \phi_e. \tag{3.1.18}$$

Proof. Suppose $\phi \in \mathcal{G}APW$. From a reformulation of Bohr's theorem to the class $\mathcal{G}APW$ (see [12, Theorem 8.11]), there exists a function $\varphi \in APW$ such that

$$\phi(x) = e^{ik(\phi)x} e^{\varphi(x)} \tag{3.1.19}$$

(and recalling that $k(\phi) \in \mathbb{R}$). Since $\varphi \in APW$, φ can be represented in the form of an absolutely convergent series

$$\varphi(x) = \sum_j \varphi_j e^{i\lambda_j x}, \quad x \in \mathbb{R}, \tag{3.1.20}$$

where $\lambda_j \in \mathbb{R}$, $\varphi_j = M(\varphi\, e_{-\lambda_j})$, and $\sum_j |\varphi_j| < \infty$. Hence, we may write

$$\varphi(x) = \sum_{\lambda_j<0} \varphi_j e^{i\lambda_j x} + \sum_{\lambda_j\geq 0} \varphi_j e^{i\lambda_j x}, \quad x \in \mathbb{R}, \tag{3.1.21}$$

where $\sum_{\lambda_j<0} \varphi_j e^{i\lambda_j x} \in APW^-$ and $\sum_{\lambda_j\geq 0} \varphi_j e^{i\lambda_j x} \in APW^+$. From (3.1.21), it follows that

$$\begin{aligned} \varphi(x) &= \Big(\sum_{\lambda_j<0} \varphi_j e^{i\lambda_j x} - \sum_{\lambda_j\geq 0} \varphi_j e^{-i\lambda_j x} \Big) + \Big(\sum_{\lambda_j\geq 0} \varphi_j e^{i\lambda_j x} + \sum_{\lambda_j\geq 0} \varphi_j e^{-i\lambda_j x} \Big) \\ &= \Big(\sum_{\lambda_j<0} \varphi_j e^{i\lambda_j x} - \sum_{\alpha_j\leq 0} \varphi_j e^{i\alpha_j x} \Big) + \sum_{\lambda_j\geq 0} \varphi_j (e^{i\lambda_j x} + e^{-i\lambda_j x}), \quad x \in \mathbb{R}, \end{aligned} \tag{3.1.22}$$

with $\alpha_j = -\lambda_j$, for all j such that $\lambda_j \geq 0$. Considering

$$\varphi_-(x) = \sum_{\lambda_j<0} \varphi_j e^{i\lambda_j x} - \sum_{\alpha_j \leq 0} \varphi_j e^{i\alpha_j x}\,, \quad x \in \mathbb{R}, \tag{3.1.23}$$

and

$$\varphi_e(x) = \sum_{\lambda_j \geq 0} \varphi_j (e^{i\lambda_j x} + e^{-i\lambda_j x})\,, \quad x \in \mathbb{R}, \tag{3.1.24}$$

we have

$$\varphi = \varphi_- + \varphi_e, \tag{3.1.25}$$

where $\varphi_- \in APW^-$ and $\varphi_e \in L^\infty(\mathbb{R})$ is an even function. Since $\phi(x) = e^{ik(\phi)x} e^{\varphi(x)}$ $(x \in \mathbb{R})$, it results that

$$\begin{aligned} \phi(x) &= e^{\varphi_-(x)} e^{ik(\phi)x} e^{\varphi_e(x)} \\ &= \phi_-(x) e^{ik(\phi)x} \phi_e(x)\,, \quad x \in \mathbb{R}, \end{aligned} \tag{3.1.26}$$

where $\phi_- = e^{\varphi_-} \in \mathcal{G}APW^-$ and $\phi_e = e^{\varphi_e} \in \mathcal{G}L^\infty(\mathbb{R})$ is an even function. $\square$

In the present section the APW asymmetric factorization will be related to a special case of right APW factorization, which we will call APW *antisymmetric factorization.* The notion of antisymmetric factorization was introduced by E. L. Basor and T. Ehrhardt in [6]. In this new kind of factorization - the antisymmetric factorization - a strong dependence between the left and the right factor occurs, as we may realize in the next definition.

Definition 3.1.4. *A function $\phi \in \mathcal{G}APW$ is said to admit an APW antisymmetric factorization if it can be represented in the form*

$$\phi = \phi_- \; e_{2\lambda} \, \widetilde{\phi_-^{-1}} \tag{3.1.27}$$

where $\phi_- \in \mathcal{G}APW^-$ and $\lambda \in \mathbb{R}$ (recall that $e_{2\lambda}(x) = e^{i2\lambda x}$, $x \in \mathbb{R}$).

The following proposition shows how the APW asymmetric factorization and the APW antisymmetric factorization are related. That is, ϕ has an APW asymmetric factorization if and only if $\Phi = \phi\widetilde{\phi^{-1}}$ has an APW antisymmetric factorization. Moreover, it shows how the APW antisymmetric factorization of $\phi\widetilde{\phi^{-1}}$ can be constructed in an explicit way from the APW asymmetric factorization of ϕ, and vice-versa.

Proposition 3.1.5. *Let $\phi \in \mathcal{G}APW$ and put $\Phi = \phi\widetilde{\phi^{-1}}$.*

(a) *If ϕ admits an APW asymmetric factorization, $\phi = \phi_-\ e_\lambda\ \phi_e$, then Φ admits an APW antisymmetric factorization with the same factor ϕ_- and the same index λ.*

(b) *If Φ admits an APW antisymmetric factorization, $\Phi = \psi_-\ e_{2\lambda}\ \widetilde{\psi_-^{-1}}$, then ϕ admits an APW asymmetric factorization with the same minus factor ψ_-, the same index λ and the even factor $\phi_e = e_{-\lambda}\psi_-^{-1}\phi$.*

Proof. (a) From the APW asymmetric factorization of ϕ, $\phi = \phi_-\ e_\lambda\ \phi_e$, we have $\widetilde{\phi^{-1}} = \phi_e^{-1}\ e_\lambda\ \widetilde{\phi_-^{-1}}$, with $\phi_- \in \mathcal{G}APW^-$. Hence $\Phi = \phi\widetilde{\phi^{-1}} = \phi_-\ e_{2\lambda}\ \widetilde{\phi_-^{-1}}$.

(b) It follows from the definition of the factor ϕ_e that $\phi = \psi_-\ e_\lambda\ \phi_e$. Thus it only remains to prove that ϕ_e is an even function. Once again by the definition of ϕ_e, we obtain

$$\widetilde{\phi_e} = e_\lambda\widetilde{\psi_-^{-1}}\widetilde{\phi} = e_\lambda e_{-2\lambda}\psi_-^{-1}\phi = e_{-\lambda}\psi_-^{-1}\phi = \phi_e, \tag{3.1.28}$$

since $\widetilde{\psi_-^{-1}}\widetilde{\phi} = e_{-2\lambda}\psi_-^{-1}\phi$ (due to the APW antisymmetric factorization of Φ). Therefore ϕ_e is an even function. □

Since APW antisymmetric factorization is a special case of right APW factorization and, by Proposition 3.1.5, we already know how to relate the APW antisymmetric factorization with the APW asymmetric factorization, it is natural to wonder how APW asymmetric factorization and right APW factorization are related. The next theorem gives the answer to this question.

Theorem 3.1.6. *Let $\phi \in \mathcal{G}APW$. If ϕ admits a right APW factorization,*

$$\phi = \varphi_- e_\lambda \varphi_+, \tag{3.1.29}$$

then ϕ admits an APW asymmetric factorization,

$$\phi = \phi_- e_\lambda \phi_e \tag{3.1.30}$$

with $\phi_- = \varphi_-\widetilde{\varphi_+^{-1}}$, and $\phi_e = \widetilde{\varphi_+}\varphi_+$.

Proof. Suppose that ϕ admits a right APW factorization, i.e, $\phi = \varphi_- e_\lambda \varphi_+$, where $\varphi_- \in \mathcal{G}APW^-$ and $\varphi_+ \in \mathcal{G}APW^+$. Considering $\Phi = \phi\widetilde{\phi^{-1}}$, we have

$$\Phi = \varphi_- e_\lambda \varphi_+ \widetilde{\varphi_+^{-1}} e_\lambda \widetilde{\varphi_-^{-1}} = \varphi_- \widetilde{\varphi_+^{-1}} e_{2\lambda} \varphi_+ \widetilde{\varphi_-^{-1}}. \tag{3.1.31}$$

Since $\varphi_- \in \mathcal{G}APW^-$ and $\varphi_+ \in \mathcal{G}APW^+$, then $\widetilde{\varphi_+^{-1}} \in \mathcal{G}APW^-$, $\widetilde{\varphi_-^{-1}} \in \mathcal{G}APW^+$, and therefore $\varphi_-\widetilde{\varphi_+^{-1}} \in \mathcal{G}APW^-$ and $\varphi_+\widetilde{\varphi_-^{-1}} \in \mathcal{G}APW^+$. Putting $\phi_- = \varphi_-\widetilde{\varphi_+^{-1}}$, it follows from (3.1.31) that $\Phi = \phi_- e_{2\lambda}\widetilde{\phi_-^{-1}}$. Since $\phi_- \in \mathcal{G}APW^-$, it results that Φ admits an APW antisymmetric factorization. By Proposition 3.1.5, this implies that ϕ admits an APW asymmetric factorization, $\phi = \phi_- e_\lambda \phi_e$, with $\phi_e = e_{-\lambda}\phi_-^{-1}\phi$. Recalling that $\phi_- = \varphi_-\widetilde{\varphi_+^{-1}}$, we may rewrite ϕ_e using the factors of the right APW factorization of ϕ, φ_- and φ_+, obtaining $\phi_e = \widetilde{\varphi_+}\varphi_+$. □

Corollary 3.1.7. *Let $\phi \in \mathcal{G}APW$. If ϕ admits a canonical right APW factorization,*

$$\phi = \varphi_- \varphi_+, \tag{3.1.32}$$

then ϕ admits a canonical APW asymmetric factorization,

$$\phi = \phi_- \phi_e \tag{3.1.33}$$

with $\phi_- = \varphi_-\widetilde{\varphi_+^{-1}}$, and $\phi_e = \widetilde{\varphi_+}\varphi_+$.

Proof. The result is a direct consequence of Theorem 3.1.6, if we take there $\lambda = 0$. □

After we have stated and proved all the results concerning to APW factorizations that will be needed to establish the invertibility and semi-Fredholm criterion for Wiener-Hopf plus Hankel operators with APW Fourier symbols, we will now extend these results for the case of almost periodic functions.

Definition 3.1.8. *We will say that a function $\phi \in \mathcal{G}AP$ admits an AP asymmetric factorization if it can be represented in the form*

$$\phi = \phi_- \, e_\lambda \, \phi_e \tag{3.1.34}$$

where $\phi_- \in \mathcal{G}AP^-$, $\phi_e \in \mathcal{G}L^\infty(\mathbb{R})$, $\widetilde{\phi_e} = \phi_e$ *and* $\lambda \in \mathbb{R}$ *(recall that* $e_\lambda(x) = e^{i\lambda x}$, $x \in \mathbb{R}$*). The particular case of an AP asymmetric factorization with* $\lambda = 0$ *will be referred to as a canonical AP asymmetric factorization.*

Example 3.1.9. Consider the function ϕ given by

$$\phi(x) = e^{-i2x + e^{i\pi x}} \ln\left(\left(\arctan(100x^2) + \frac{\pi}{2}\right)^{e^{-i2\sin(\pi x)}}\right), \quad x \in \mathbb{R}. \tag{3.1.35}$$

Clearly, ϕ is an almost periodic function, as it can be confirmed by the images of $\phi(x)$ (when x belongs to four different intervals) exhibit in Figure 3.1. We may rewrite ϕ as

$$\phi(x) = e^{e^{-i\pi x}} e^{-i2x} \ln\left(\arctan(100x^2) + \frac{\pi}{2}\right), \quad x \in \mathbb{R}. \tag{3.1.36}$$

Considering ϕ_- and ϕ_e given by

$$\phi_-(x) = e^{e^{-i\pi x}}, \quad x \in \mathbb{R}, \tag{3.1.37}$$

$$\phi_e(x) = \ln\left(\arctan(100x^2) + \frac{\pi}{2}\right), \quad x \in \mathbb{R}, \tag{3.1.38}$$

we have $\phi_- \in \mathcal{G}AP^-$ and $\phi_e \in \mathcal{G}L^\infty(\mathbb{R})$ such that $\widetilde{\phi_e} = \phi_e$. Therefore, it follows that ϕ admits an AP asymmetric factorization.

As about more examples, in APW we find an endless number of almost periodic functions which have an AP asymmetric factorization. Indeed, recalling that in Theorem 3.1.3 it was proved that every invertible APW function admits an APW asymmetric factorization and due to the fact that AP asymmetric factorization is a generalization of APW asymmetric factorization, it results that every invertible APW function admits an AP asymmetric factorization.

Similarly to the APW asymmetric factorization, the AP asymmetric factorization of an almost periodic function, when exists, is also unique up to a constant, like it is stated in the following proposition.

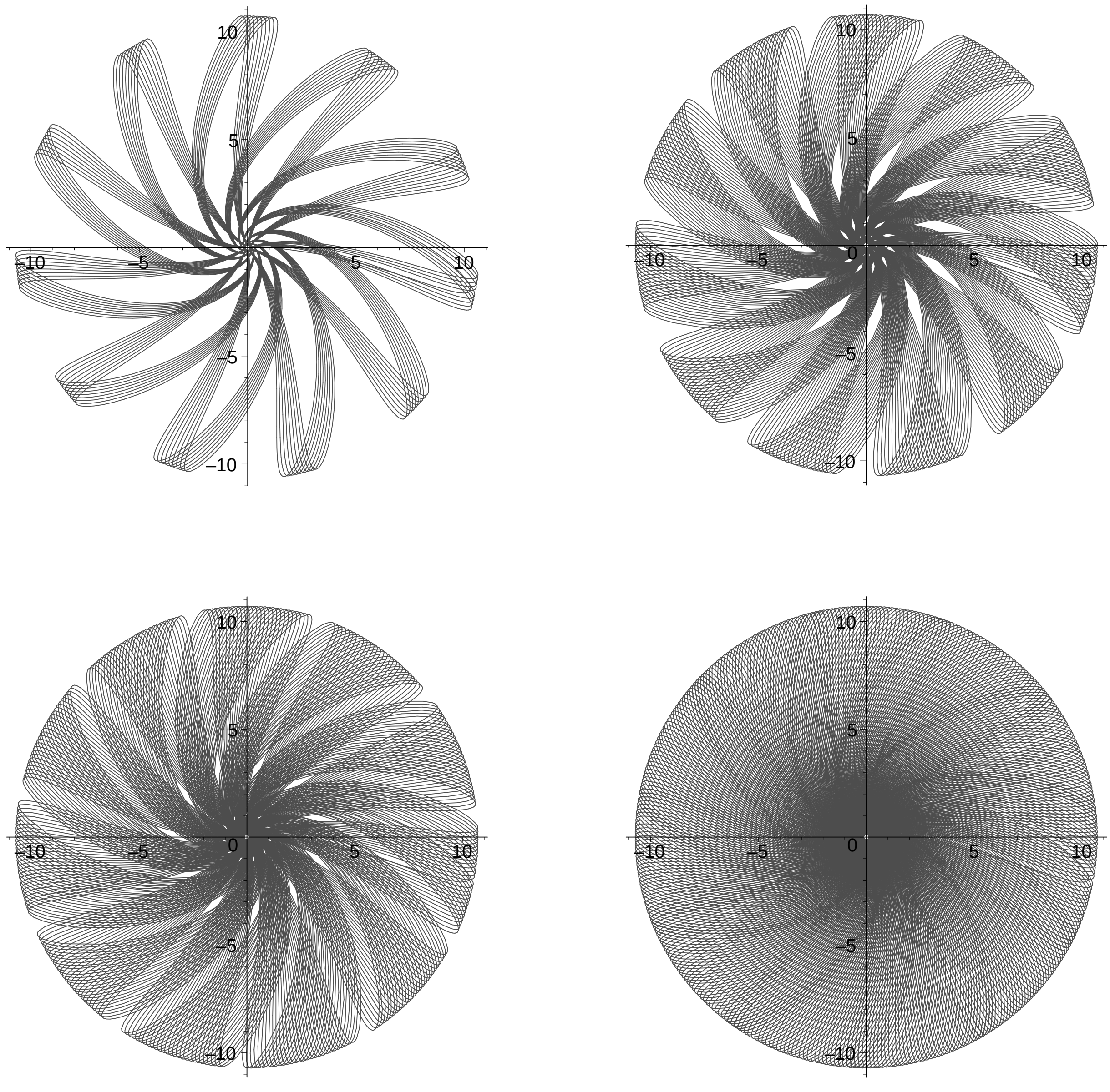


Figure 3.1: The image of $\phi(x) = e^{-i2x+e^{i\pi x}} \ln\left((\arctan(100x^2) + \frac{\pi}{2})^{e^{-i2\sin(\pi x)}}\right)$ for x in $[-100, 100]$, $[-250, 250]$, $[-300, 300]$ and $[-400, 400]$, respectively.

Proposition 3.1.10. *Let $\phi \in \mathcal{G}AP$. Suppose that ϕ admits two AP asymmetric factorizations:*

$$\phi = \phi_-^{(1)} \, e_{\lambda_1} \, \phi_e^{(1)} \,, \tag{3.1.39}$$

$$\phi = \phi_-^{(2)} \, e_{\lambda_2} \, \phi_e^{(2)} \,. \tag{3.1.40}$$

Then $\lambda_1 = \lambda_2$, $\phi_-^{(1)} = \gamma\phi_-^{(2)}$ and $\phi_e^{(1)} = \gamma^{-1}\phi_e^{(2)}$, $\gamma \in \mathbb{C} \setminus \{0\}$.

Such as in the APW asymmetric factorization case, it is also possible to relate the AP asymmetric factorization with a special case of right AP factorization. In this special case of right AP factorization a strong dependence between the left and the right factor occurs and due to that it will be called AP *antisymmetric factorization*.

Definition 3.1.11. *A function $\phi \in \mathcal{G}AP$ is said to admit an AP antisymmetric factorization if it can be represented in the form*

$$\phi = \phi_- \, e_{2\lambda} \, \widetilde{\phi_-^{-1}} \tag{3.1.41}$$

where $\phi_- \in \mathcal{G}AP^-$ and $\lambda \in \mathbb{R}$ (recall that $e_{2\lambda}(x) = e^{i2\lambda x}$, $x \in \mathbb{R}$).

Analogously to the APW asymmetric factorization, in this case it holds the same relation between the AP asymmetric factorization and the AP antisymmetric factorization, i.e., ϕ has an AP asymmetric factorization if and only if $\Phi = \phi\widetilde{\phi^{-1}}$ has an AP antisymmetric factorization.

Proposition 3.1.12. *Let $\phi \in \mathcal{G}AP$ and put $\Phi = \phi\widetilde{\phi^{-1}}$.*

(a) *If ϕ admits an AP asymmetric factorization, $\phi = \phi_- \, e_\lambda \, \phi_e$, then Φ admits an AP antisymmetric factorization with the same factor ϕ_- and the same index λ.*

(b) *If Φ admits an AP antisymmetric factorization, $\Phi = \psi_- \, e_{2\lambda} \, \widetilde{\psi_-^{-1}}$, then ϕ admits an AP asymmetric factorization with the same minus factor ψ_-, the same index λ and the even factor $\phi_e = e_{-\lambda}\psi_-^{-1}\phi$.*

As a consequence of Proposition 3.1.12, and recalling that AP antisymmetric factorization is a special case of right AP factorization, we provide a way to relate the right AP factorization with the AP asymmetric factorization.

Theorem 3.1.13. *Let $\phi \in \mathcal{G}AP$. If ϕ admits a right AP factorization,*

$$\phi = \varphi_- e_\lambda \varphi_+, \tag{3.1.42}$$

then ϕ admits an AP asymmetric factorization,

$$\phi = \phi_- e_\lambda \phi_e \tag{3.1.43}$$

with $\phi_- = \varphi_- \widetilde{\varphi_+^{-1}}$, and $\phi_e = \widetilde{\varphi_+}\varphi_+$.

Corollary 3.1.14. *Let $\phi \in \mathcal{G}AP$. If ϕ admits a canonical right AP factorization,*

$$\phi = \varphi_- \varphi_+, \tag{3.1.44}$$

then ϕ admits a canonical AP asymmetric factorization,

$$\phi = \phi_- \phi_e \tag{3.1.45}$$

with $\phi_- = \varphi_- \widetilde{\varphi_+^{-1}}$, and $\phi_e = \widetilde{\varphi_+}\varphi_+$.

3.2 Invertibility of Wiener-Hopf plus Hankel operators with APW symbols

In this section, we present an invertibility and semi-Fredholm criterion, as well as an explicit formula for the (one-sided and two-sided) inverses of Wiener-Hopf plus Hankel operators with APW Fourier symbols.

Theorem 3.2.1. *Let $\phi \in \mathcal{G}APW$.*

(a) *If $k(\phi) < 0$, then $(W+H)_\phi$ is properly d-normal and right-invertible.*

(b) *If $k(\phi) > 0$, then $(W+H)_\phi$ is properly n-normal and left-invertible.*

(c) *If $k(\phi) = 0$, then $(W+H)_\phi$ is invertible.*

Therefore, if $(W+H)_\phi$ is a Fredholm operator then $(W+H)_\phi$ is invertible.

Proof. From Theorem 3.1.3 it follows that ϕ admits an APW asymmetric factorization of the form

$$\phi = \phi_- \, e_{k(\phi)} \, \phi_e. \tag{3.2.46}$$

In the case where $k(\phi) < 0$, we have that $e_{k(\phi)} \in AP^-$. Since $AP^- = AP \cap H^\infty_-(\mathbb{R})$, it holds that $e_{k(\phi)} \in H^\infty_-(\mathbb{R})$ and hence

$$(W+H)_\phi = W_{\phi_-} \, \ell_0 \, W_{e_{k(\phi)}} \, \ell_0 \, (W+H)_{\phi_e}, \tag{3.2.47}$$

due to Proposition 1.3.4 and (1.3.61). Since $\phi_- \in \mathcal{G}APW^-$, by the characterization of $\mathcal{G}APW^-$, there exists a function $\psi \in APW^-$ such that $\phi_- = e^\psi$. Thus, the mean motion of ϕ_- is zero and by Theorem 3.0.3, W_{ϕ_-} is invertible. From Proposition 1.3.5, we know that $(W+H)_{\phi_e}$ is invertible. Therefore, since $\ell_0 : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R})$ is also an invertible operator, (3.2.47) shows that $(W+H)_\phi$ is equivalent to $W_{e_{k(\phi)}}$. Once again, by Theorem 3.0.3, since the mean motion of $e_{k(\phi)}$ is $k(\phi)$ and $k(\phi) < 0$, it follows that the operator $W_{e_{k(\phi)}}$ is properly d-normal and right-invertible. Consequently, due to the equivalence relation (3.2.47), the operator $(W+H)_\phi$ is also properly d-normal and right-invertible. This completes the proof of part (a).

Part (b) can be derived from part (a) by passage to adjoint operators.

Finally, let us now suppose that $k(\phi) = 0$. Then $\phi = \phi_- \, \phi_e$ and

$$(W+H)_\phi = W_{\phi_-} \, \ell_0 \, (W+H)_{\phi_e}. \tag{3.2.48}$$

Since W_{ϕ_-} and $(W+H)_{\phi_e}$ are invertible, then $(W+H)_\phi$ is also invertible, and therefore the proof of part (c) is complete. $\square$

In order to proceed to the next result, we recall here that T is reflexive generalized invertible if and only if $\operatorname{Ker} T$ is a complementable subspace of X and $\operatorname{Im} T$ is a closed and complementable subspace of Y. Consequently, each Fredholm operator is reflexive generalized invertible. Therefore and after having reached an invertibility and semi-Fredholm criterion for the Wiener-Hopf plus Hankel operators with APW symbols, we start now looking for an explicit formula for the reflexive generalized inverses of these Wiener-Hopf plus Hankel operators.

The obtained formula exhibits the importance of the APW asymmetric factorization of the Fourier symbols of the Wiener-Hopf plus Hankel operators since the inverses are expressed in terms of the factors of the APW asymmetric factorization and the value of the mean motion of ϕ allows to distinguish if the reflexive generalized inverse is in fact a right-inverse, left-inverse or inverse. I.e., depending on $\kappa(\phi) < 0$, $\kappa(\phi) > 0$ and $\kappa(\phi) = 0$, we obtain the right-inverse, the left-inverse and the inverse of $(W+H)_\phi$, respectively. As we will see in the theorem below, the explicit formula for the reflexive generalized inverse of $(W+H)_\phi$ is given in terms of an arbitrary extension operator $\ell : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$. This means that the reflexive generalized inverse of $(W+H)_\phi$ is independent of the choice of ℓ (and therefore several choices are allowed, like for instance $\ell = \ell_0$ or $\ell = \ell^e$).

Theorem 3.2.2. *Let $\phi \in \mathcal{G}APW$ and*

$$(W+H)^-_\phi = \ell_0 r_+ \mathcal{F}^{-1}\phi_e^{-1} \cdot \mathcal{F}\ell^e r_+ \mathcal{F}^{-1} e_{-k(\phi)} \cdot \mathcal{F}\ell^e r_+ \mathcal{F}^{-1}\phi_-^{-1} \cdot \mathcal{F}\ell : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R}), \quad (3.2.49)$$

where ϕ_e^{-1}, $e_{-k(\phi)}$, and ϕ_-^{-1} are the inverses of the corresponding factors of an APW asymmetric factorization of ϕ, $\phi = \phi_-\, e_{k(\phi)}\, \phi_e$, and the operator $\ell : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ denotes an arbitrary extension operator. Then the operator $(W+H)^-_\phi$ is a reflexive generalized inverse of $(W+H)_\phi$ and, moreover, it is

(a) *the right-inverse of $(W+H)_\phi$, if $k(\phi) < 0$,*

(b) *the left-inverse of $(W+H)_\phi$, if $k(\phi) > 0$,*

(c) *the inverse of $(W+H)_\phi$, if $k(\phi) = 0$.*

Proof. Since $\phi \in \mathcal{G}APW$, due to Theorem 3.1.3, we have that ϕ admits an APW asymmetric factorization of the form

$$\phi = \phi_-\, e_{k(\phi)}\, \phi_e. \quad (3.2.50)$$

Consequently, from (1.1.32), it follows that

$$(W+H)_\phi = r_+ A_- E A_e \ell^e r_+, \tag{3.2.51}$$

where

$$A_- = \mathcal{F}^{-1}\phi_- \cdot \mathcal{F}, \tag{3.2.52}$$

$$E = \mathcal{F}^{-1} e_{k(\phi)} \cdot \mathcal{F}, \tag{3.2.53}$$

$$A_e = \mathcal{F}^{-1}\phi_e \cdot \mathcal{F}. \tag{3.2.54}$$

(i) If $k(\phi) \leq 0$, consider

$$\begin{aligned}(W+H)_\phi \, (W+H)_\phi^- &= r_+ A_- E A_e \ell^e r_+ \; \ell_0 r_+ A_e^{-1} \ell^e r_+ E^{-1} \ell^e r_+ A_-^{-1} \ell \\ &= r_+ A_- E A_e \ell^e r_+ A_e^{-1} \ell^e r_+ E^{-1} \ell^e r_+ A_-^{-1} \ell, \end{aligned} \tag{3.2.55}$$

where the term $\ell_0 r_+$ was omitted due to the fact that $r_+ \ell_0 r_+ = r_+$. Since A_e^{-1} preserves the even property of its symbol, we may also drop the first $\ell^e r_+$ term in (3.2.55), and obtain

$$(W+H)_\phi \, (W+H)_\phi^- = r_+ A_- E \ell^e r_+ E^{-1} \ell^e r_+ A_-^{-1} \ell. \tag{3.2.56}$$

Additionally, since in the present case E is a *minus type factor*, i.e., $k(\phi) \leq 0$ in (3.2.53) (see [22, 73]), it follows that E^{-1} is a *plus type factor*, i.e., $\lambda \geq 0$ in the representation of E^{-1} as $E^{-1} = \mathcal{F}^{-1} e_\lambda \cdot \mathcal{F}$. Due to this and to the properties of A_-, we may also omit the first $\ell^e r_+$ term in (3.2.56). It follows then that

$$(W+H)_\phi \, (W+H)_\phi^- = r_+ A_- \ell^e r_+ A_-^{-1} \ell = r_+ \ell = I_{L^2(\mathbb{R}_+)}. \tag{3.2.57}$$

(ii) If $k(\phi) \geq 0$, we will now analyze the composition

$$(W+H)_\phi^- \, (W+H)_\phi = \ell_0 r_+ A_e^{-1} \ell^e r_+ E^{-1} \ell^e r_+ A_-^{-1} \ell \; r_+ A_- E A_e \ell^e r_+ \, . \tag{3.2.58}$$

In the present case E^{-1} is a *minus type factor* and for this reason $\ell^e r_+ E^{-1} \ell^e r_+ = \ell^e r_+ E^{-1}$. The same reasoning applies to the factor A_-^{-1}, and therefore the equality (3.2.58) takes the form

$$(W+H)_\phi^- \, (W+H)_\phi = \ell_0 r_+ A_e^{-1} \ell^e r_+ A_e \ell^e r_+ = \ell_0 r_+ \ell^e r_+ = \ell_0 r_+ = I_{L^2_+(\mathbb{R})} \, , \tag{3.2.59}$$

where we took into account that $\ell^e r_+ A_e \ell^e r_+ = A_e \ell^e r_+$.

(iii) Intersecting the last two cases, (i) and (ii), it follows that for $k(\phi) = 0$, the operator $(W+H)_\phi^-$ is the (both-sided) inverse of $(W+H)_\phi$ (cf. (3.2.57) and (3.2.59)).

Finally, please observe that all the above three situations are stronger than

$$(W+H)_\phi \, (W+H)_\phi^- \, (W+H)_\phi = (W+H)_\phi \,, \quad (W+H)_\phi^- \, (W+H)_\phi \, (W+H)_\phi^- = (W+H)_\phi^- \,, \tag{3.2.60}$$

i.e., $(W+H)_\phi^-$ is a reflexive generalized inverse of $(W+H)_\phi$. □

As a consequence of Theorem 3.2.1 and of the relation between right APW factorization and APW asymmetric factorization presented in Theorem 3.1.6 and Corollary 3.1.7, we end up this section with a curious result about the dependence between the invertibility of Wiener-Hopf and Wiener-Hopf plus Hankel operators with the same APW Fourier symbol. That is, from the invertibility of the Wiener-Hopf operator, we obtain the invertibility of the Wiener-Hopf plus Hankel operator.

Theorem 3.2.3. *Let $\phi \in \mathcal{G}APW$. If W_ϕ is invertible, then $(W+H)_\phi$ is invertible.*

Proof. According to [12, Corollary 9.8], if $\phi \in \mathcal{G}APW$, then W_ϕ is invertible if and only if ϕ admits a canonical right APW factorization. Thus, since W_ϕ is invertible, ϕ admits a canonical right APW factorization. Suppose that

$$\phi = \varphi_- \varphi_+ \tag{3.2.61}$$

is a canonical right APW factorization of ϕ. By Corollary 3.1.7, ϕ admits a canonical APW asymmetric factorization,

$$\phi = \phi_- \phi_e , \tag{3.2.62}$$

where

$$\phi_- = \varphi_- \, \widetilde{\varphi_+^{-1}} \,, \qquad \phi_e = \widetilde{\varphi_+} \, \varphi_+ \,. \tag{3.2.63}$$

From Theorem 3.2.1, it follows that $(W+H)_\phi$ is an invertible operator. □

3.3 Invertibility of Wiener-Hopf plus Hankel operators with AP symbols

After we have achieved an invertibility and semi-Fredholm criterion, as well as an explicit formula for the (one-sided and two-sided) inverses of Wiener-Hopf plus Hankel operators with APW Fourier symbols, in this section we will consider the general case of Wiener-Hopf plus Hankel operators with almost periodic Fourier symbols. As mentioned before, there is a result in Section 3.1 that it is not possible to generalize from the APW case to the AP case. That is, we cannot guarantee that every invertible AP function admits an AP asymmetric factorization, while in Theorem 3.1.3 we proved that every invertible APW function admits an APW asymmetric factorization. Due to this, the results presented in this section are concerned with the invertibility and semi-Fredholm property and to the one-sided and two-sided inverses of the operators are obtained assuming an AP asymmetric factorization of the Fourier symbol of the operators in study.

We omit the proofs in this section since the results that will be presented next have similar proofs of the analogous APW results.

Theorem 3.3.1. *Let $\phi \in \mathcal{G}AP$ admit an AP asymmetric factorization $\phi = \phi_- \, e_\lambda \, \phi_e$.*

(a) *If $\lambda < 0$, then $(W+H)_\phi$ is properly d-normal and right-invertible.*

(b) *If $\lambda > 0$, then $(W+H)_\phi$ is properly n-normal and left-invertible.*

(c) *If $\lambda = 0$, then $(W+H)_\phi$ is invertible.*

Therefore, if $(W+H)_\phi$ is a Fredholm operator then $(W+H)_\phi$ is invertible.

Theorem 3.3.2. *If $\phi \in \mathcal{G}AP$ admits an AP asymmetric factorization*

$$\phi = \phi_- \, e_\lambda \, \phi_e \,, \tag{3.3.64}$$

then a reflexive generalized inverse of $(W+H)_\phi$ is defined by

$$(W+H)_\phi^- = \ell_0 r_+ \mathcal{F}^{-1} \phi_e^{-1} \cdot \mathcal{F} \ell^e r_+ \mathcal{F}^{-1} e_{-\lambda} \cdot \mathcal{F} \ell^e r_+ \mathcal{F}^{-1} \phi_-^{-1} \cdot \mathcal{F} \ell : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R}), \tag{3.3.65}$$

where $\ell : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ *denotes an arbitrary extension operator.*

Additionally, in a more detailed way:

(a) *if* $\lambda < 0$, *then* $(W+H)^-_\phi$ *is the right-inverse of* $(W+H)_\phi$*;*

(b) *if* $\lambda > 0$, *then* $(W+H)^-_\phi$ *is the left-inverse of* $(W+H)_\phi$*;*

(c) *if* $\lambda = 0$, *then* $(W+H)^-_\phi$ *is the inverse of* $(W+H)_\phi$*.*

Combining now Theorem 3.3.1, and Theorem 3.1.13 and Corollary 3.1.14 (concerning to the relation between right AP factorization and AP asymmetric factorization), we can also establish invertibility dependencies between Wiener-Hopf and Wiener-Hopf plus Hankel operators with the same AP Fourier symbol. In this case, we also have to deal with an additional assumption, this time not concerning to the AP asymmetric factorization of the Fourier symbols of the operators, but concerning with the right AP factorization. This is due to the AP analogue of Simonenko's theorem that states that, for $\phi \in AP$, W_ϕ is invertible if and only if ϕ admits a canonical generalized right AP factorization (see [12, Theorem 21.7]). The generalized right AP factorization, as its name suggest, is a generalization of the right AP factorization, and it was introduced to prove the AP analogue of Simonenko's theorem mentioned before. We will not go into details about this new kind of AP factorization. For more information about generalized right AP factorization, see [12, 13]. Seeing that W_ϕ is invertible if and only if ϕ admits a canonical generalized right AP factorization and that we can only establish a relation between AP asymmetric factorization and right AP factorization (which is a particular case of a generalized right AP factorization), the assumption concerning to the canonical right AP factorization arises naturally in the attainment of invertibility dependencies between Wiener-Hopf and Wiener-Hopf plus Hankel operators.

Theorem 3.3.3. *Let* $\phi \in \mathcal{G}AP$*. If* W_ϕ *is invertible with* ϕ *having a canonical right AP factorization, then* $(W+H)_\phi$ *is invertible.*

Proof. Suppose that

$$\phi = \varphi_- \varphi_+ \tag{3.3.66}$$

is a canonical right AP factorization of ϕ. By Corollary 3.1.14, ϕ admits a canonical AP asymmetric factorization,

$$\phi = \phi_- \phi_e, \tag{3.3.67}$$

where

$$\phi_- = \varphi_- \, \widetilde{\varphi_+^{-1}}\,, \qquad \phi_e = \widetilde{\varphi_+}\, \varphi_+\,. \tag{3.3.68}$$

From Theorem 3.3.1, it follows that $(W+H)_\phi$ is an invertible operator. ☐

Remark 3.3.4. We would like to mention that Theorems 3.2.1, 3.2.3, 3.3.1 and 3.3.3 concerning to Wiener-Hopf plus Hankel operators also hold true for Wiener-Hopf minus Hankel operators. This occurs because for Wiener-Hopf minus Hankel operators we have an analogue of Proposition 1.3.2 and due to this we obtain a similar result of Proposition 1.3.4 (cf. Remark 1.3.6). Similar versions of Theorems 3.2.2 and 3.3.2 are also possible to obtain for Wiener-Hopf minus Hankel operators if we consider the reflexive generalized inverses of $(W-H)_\phi$ defined by

$$(W-H)_\phi^- = \ell_0 r_+ \mathcal{F}^{-1} \phi_e^{-1} \cdot \mathcal{F} \ell^o r_+ \mathcal{F}^{-1} e_{-\lambda} \cdot \mathcal{F} \ell^o r_+ \mathcal{F}^{-1} \phi_-^{-1} \cdot \mathcal{F} \ell : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R}). \tag{3.3.69}$$

Chapter 4

Wiener-Hopf-Hankel Operators with Semi-Almost Periodic Symbols

In 1977, D. Sarason [67] developed the semi-Fredholm theory for Toeplitz operators in the Hardy space H^2, with symbols in the algebra of semi-almost periodic elements. Few years later, in 1981, R. V. Duduchava and A. I. Saginashvili [29, 65] worked out the corresponding semi-Fredholm theory for Wiener-Hopf operators with semi-almost periodic Fourier symbols, acting between L^p Lebesgue spaces $(1 < p < \infty)$. Both results of D. Sarason, and R. V. Duduchava and A. I. Saginashvili were obtained upon conditions on the mean motions and on the geometric mean values of the almost periodic representatives at minus and plus infinity of the Fourier symbols.

Motivated by these results, in this chapter, we establish a corresponding analysis for Wiener-Hopf-Hankel operators with semi-almost periodic Fourier symbols. In a first part, by considering Wiener-Hopf-Hankel operators acting in L^2 Lebesgue spaces and then, in a second part, considering Wiener-Hopf-Hankel operators acting in L^p Lebesgue spaces $(1 < p < \infty)$. To end up, we dedicate a section to the study of a formula for the Fredholm index of the Wiener-Hopf-Hankel operators.

4.1 A Sarason's type theorem

4.1.1 Motivation

The aim of this section is the obtainment of a Sarason's type theorem for Wiener-Hopf--Hankel operators. We can find the first version of the Sarason's theorem in [67, Theorem 1]. There, D. Sarason only considers Toeplitz operators with Fourier symbols invertible in the algebra of semi-almost periodic functions. In [12, Theorem 3.9], the Sarason's Theorem for Wiener-Hopf operators with semi-almost periodic Fourier symbols is presented, where the case of Wiener-Hopf operators with Fourier symbols that are not invertible in the algebra of semi-almost periodic functions is also considered. This result is presented below.

From now on, consider $[c_1, c_2]$ to be the line segment in the complex plane between and including the endpoints $c_1, c_2 \in \mathbb{C}$.

Theorem 4.1.1. ([12, Theorem 3.9]) *Let $\phi \in SAP$ and suppose that ϕ is not identically zero.*

(a) *If $\phi \notin \mathcal{G}SAP$, then W_ϕ is not normally solvable.*

(b) *If $\phi \in \mathcal{G}SAP$ and $\kappa(\phi_l) \cdot \kappa(\phi_r) < 0$, then W_ϕ is not normally solvable.*

(c) *If $\phi \in \mathcal{G}SAP$, $\kappa(\phi_l) \cdot \kappa(\phi_r) \geq 0$, and $\kappa(\phi_l) + \kappa(\phi_r) > 0$, then W_ϕ is properly n-normal and left-invertible.*

(d) *If $\phi \in \mathcal{G}SAP$, $\kappa(\phi_l) \cdot \kappa(\phi_r) \geq 0$, and $\kappa(\phi_l) + \kappa(\phi_r) < 0$, then W_ϕ is properly d-normal and right-invertible.*

(e) *If $\phi \in \mathcal{G}SAP$, $\kappa(\phi_l) = \kappa(\phi_r) = 0$, and $0 \in [\mathbf{d}(\phi_l), \mathbf{d}(\phi_r)]$, then W_ϕ is not normally solvable.*

(f) *If $\phi \in \mathcal{G}SAP$, $\kappa(\phi_l) = \kappa(\phi_r) = 0$, and $0 \notin [\mathbf{d}(\phi_l), \mathbf{d}(\phi_r)]$, then W_ϕ is a Fredholm operator.*

While D. Sarason used an approach related with localization techniques and $C^\star$-algebra arguments to prove the Sarason's Theorem, A. Böttcher, Yu. I. Karlovich and I. M. Spitkovsky followed the arguments used by R. V. Duduchava and A. I. Saginashvili in [29]. Although some parts of the approach of R. V. Duduchava and A. I. Saginashvili are also possible to be carried out to Wiener-Hopf-Hankel operators, there are other parts where we have to use a different reasoning. After some attempts, we were able to provide a proof of the Sarason's type theorem in a much more simple and elegant way by using the Δ–relation after extension as it is stated in the next section. In Section 4.1.3, we follow, as far as possible, a similar approach of that one used by R. V. Duduchava and A. I. Saginashvili. Since, in this case, we obtain a different version of the Sarason's type theorem, combining these two versions, we can refine the initial version of the Sarason's type theorem. In this sense, we will end this section by presenting a stronger version of the Sarason's type theorem.

4.1.2 First approach

Theorem 4.1.2. *Let $\phi \in \mathcal{G}SAP$.*

(a) *If $\kappa(\phi_l) + \kappa(\phi_r) < 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are right-invertible. Moreover, at least one of these operators is properly d-normal.*

(b) *If $\kappa(\phi_l) + \kappa(\phi_r) > 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are left-invertible. In addition, at least one of these operators is properly n-normal.*

(c) *If $\kappa(\phi_l) + \kappa(\phi_r) = 0$ and $\Re e\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \neq 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are Fredholm operators.*

(d) *If $\kappa(\phi_l) + \kappa(\phi_r) = 0$ and $\Re e\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) = 0$, then at least one of the operators $(W+H)_\phi$ and $(W-H)_\phi$ is not normally solvable.*

Proof. By the well-known characterization of SAP (cf. (2.2.27)), considering $u \in C(\overline{\mathbb{R}})$

for which $u(-\infty) = 0$ and $u(+\infty) = 1$, there exist $\phi_l, \phi_r \in AP$ and $\phi_0 \in C_0(\dot{\mathbb{R}})$ such that

$$\phi = (1-u)\phi_l + u\phi_r + \phi_0. \tag{4.1.1}$$

Because $\phi \in \mathcal{G}SAP$, it results that $\phi_l, \phi_r \in \mathcal{G}AP$. Consequently, taking into consideration Bohr's Theorem and the definition of the geometric mean value (cf. (2.1.13) and (2.1.15)), it follows that

$$\phi_l = e_{\kappa(\phi_l)}\mathbf{d}(\phi_l)e^{\omega_l} \tag{4.1.2}$$

and

$$\phi_r = e_{\kappa(\phi_r)}\mathbf{d}(\phi_r)e^{\omega_r} \tag{4.1.3}$$

with $\omega_l, \omega_r \in AP$ and $M(\omega_l) = M(\omega_r) = 0$ (and obviously $\mathbf{d}(\phi_l)\mathbf{d}(\phi_r) \neq 0$). Thus

$$\phi = (1-u)\mathbf{d}(\phi_l)e_{\kappa(\phi_l)}e^{\omega_l} + u\mathbf{d}(\phi_r)e_{\kappa(\phi_r)}e^{\omega_r} + \phi_0. \tag{4.1.4}$$

In view of the transfer of regularity properties from the Wiener-Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ to the Wiener-Hopf plus Hankel operator $(W+H)_\phi$ and to the Wiener-Hopf minus Hankel operator $(W-H)_\phi$ (stated in Corollary 1.3.8 and Corollary 1.3.10), we will now study the Wiener-Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$. Since

$$\widetilde{\phi} = (1-\widetilde{u})\mathbf{d}(\phi_l)e_{-\kappa(\phi_l)}e^{\widetilde{\omega_l}} + \widetilde{u}\mathbf{d}(\phi_r)e_{-\kappa(\phi_r)}e^{\widetilde{\omega_r}} + \widetilde{\phi}_0, \tag{4.1.5}$$

we have

$$\phi\widetilde{\phi^{-1}} = \frac{(1-u)\mathbf{d}(\phi_l)e_{\kappa(\phi_l)}e^{\omega_l} + u\mathbf{d}(\phi_r)e_{\kappa(\phi_r)}e^{\omega_r} + \phi_0}{(1-\widetilde{u})\mathbf{d}(\phi_l)e_{-\kappa(\phi_l)}e^{\widetilde{\omega_l}} + \widetilde{u}\mathbf{d}(\phi_r)e_{-\kappa(\phi_r)}e^{\widetilde{\omega_r}} + \widetilde{\phi_0}}. \tag{4.1.6}$$

To find the almost periodic representatives of $\phi\widetilde{\phi^{-1}}$, we just have to study the behaviour of $\phi\widetilde{\phi^{-1}}$ at $-\infty$ and $+\infty$ (recall that the almost periodic representatives of an almost periodic function are uniquely determined). Therefore, for the almost periodic representatives of $\phi\widetilde{\phi^{-1}}$, we get

$$\left(\phi\widetilde{\phi^{-1}}\right)_l = \frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\, e_{\kappa(\phi_l)+\kappa(\phi_r)}e^{\omega_l-\widetilde{\omega_r}}\,, \tag{4.1.7}$$

$$\left(\phi\widetilde{\phi^{-1}}\right)_r = \frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}\, e_{\kappa(\phi_l)+\kappa(\phi_r)}\, e^{\omega_r-\widetilde{\omega_l}}\,. \tag{4.1.8}$$

Because $\omega_l, \omega_r \in AP$ are such that $M(\omega_l) = M(\omega_r) = 0$ (which additionally implies $M(\widetilde{\omega_l}) = M(\widetilde{\omega_r}) = 0$), it follows from (4.1.7) and (4.1.8) that

$$\kappa\Big(\big(\phi\widetilde{\phi^{-1}}\big)_l\Big) = \kappa\Big(\big(\phi\widetilde{\phi^{-1}}\big)_r\Big) = \kappa(\phi_l) + \kappa(\phi_r) \tag{4.1.9}$$

and

$$\mathbf{d}\Big(\big(\phi\widetilde{\phi^{-1}}\big)_l\Big) = \frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}, \qquad \mathbf{d}\Big(\big(\phi\widetilde{\phi^{-1}}\big)_r\Big) = \frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}. \tag{4.1.10}$$

Applying now Sarason's Theorem to the Wiener-Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$, it holds that:

(a) if $\kappa(\phi_l) + \kappa(\phi_r) < 0$, then $W_{\phi\widetilde{\phi^{-1}}}$ is properly d-normal and right-invertible;

(b) if $\kappa(\phi_l) + \kappa(\phi_r) > 0$, then $W_{\phi\widetilde{\phi^{-1}}}$ is properly n-normal and left-invertible;

(c) if $\kappa(\phi_l) + \kappa(\phi_r) = 0$ and

$$0 \notin \Big[\mathbf{d}\Big(\big(\phi\widetilde{\phi^{-1}}\big)_l\Big),\ \mathbf{d}\Big(\big(\phi\widetilde{\phi^{-1}}\big)_r\Big)\Big], \tag{4.1.11}$$

then $W_{\phi\widetilde{\phi^{-1}}}$ is a Fredholm operator;

(d) if $\kappa(\phi_l)+\kappa(\phi_r) = 0$ and (4.1.11) does not happen, then $W_{\phi\widetilde{\phi^{-1}}}$ is not normally solvable.

In this case, since $\mathbf{d}\left((\phi\widetilde{\phi^{-1}})_l\right)$ and $\mathbf{d}\left((\phi\widetilde{\phi^{-1}})_r\right)$ are inverses of each other (cf. (4.1.10)), the condition (4.1.11) is satisfied if and only if $\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)$ is such that

$$\Re e\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \neq 0. \tag{4.1.12}$$

Applying now Corollary 1.3.8 and Corollary 1.3.10, we obtain the following: $(W+H)_\phi$ and $(W-H)_\phi$ are right-invertible, under the conditions of case (a); $(W+H)_\phi$ and $(W-H)_\phi$ are left-invertible, under the conditions of case (b); and $(W+H)_\phi$ and $(W-H)_\phi$ are Fredholm operators, under the conditions of case (c). To arrive at the final assertion, we can interpret the Δ–relation after extension between the Wiener-Hopf plus Hankel operator $(W+H)_\phi$ and the Wiener-Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ (presented in Lemma 1.3.7) as an equivalence after extension between $\mathrm{diag}[(W+H)_\phi, \mathcal{T}_\phi]$ and $W_{\phi\widetilde{\phi^{-1}}}$. In this way, we get in cases (a) and (b) that $\mathrm{diag}[(W+H)_\phi, \mathcal{T}_\phi]$ is properly d-normal or properly n-normal, respectively. This means that at least one of the operators $(W+H)_\phi$ and $\mathcal{T}_\phi$ is properly d-normal or properly n-normal, in the case (a) or (b), respectively. Considering now the equivalence after extension between $\mathcal{T}_\phi$ and $(W-H)_\phi$ (cf. Proposition 1.3.9), the last proposition tells us that at least one the operators $(W+H)_\phi$ and $(W-H)_\phi$ is properly d-normal or properly n-normal, if in the conditions of case (a) or case (b), respectively. In case (d), we have that $\operatorname{diag}[(W+H)_\phi, \mathcal{T}_\phi]$ is not normally solvable, which implies that at least one the operators $(W+H)_\phi$ and $\mathcal{T}_\phi$ is not normally solvable. From the equivalence after extension (1.3.109), it therefore follows that at least one the operators $(W+H)_\phi$ and $(W-H)_\phi$ is not normally solvable (in this case (d)). $\square$

Remark 4.1.3. We would like to point out the following details concerning to this last result:

(i) Theorem 4.1.2 is called a Sarason's type theorem for Wiener-Hopf-Hankel operators since it describes the Fredholm nature of $(W+H)_\phi$ and $(W-H)_\phi$ based on the values of $\kappa(\phi_l)$, $\kappa(\phi_r)$, $\mathbf{d}(\phi_l)$ and $\mathbf{d}(\phi_r)$ when $\phi \in \mathcal{G}SAP$;

(ii) being clear that Theorem 4.1.2 is not a complete characterization of all the regularity properties of $(W \pm H)_\phi$, we would like to mentioned that under the conditions of the theorem some open questions remain to be answered. For instance, this is the case of condition (d) which is formulated as a minimal condition about the image of the two operators;

(iii) finally, from the above theorem and from the Sarason's Theorem for Wiener-Hopf operators W_ϕ, it follows that if $\phi \in \mathcal{G}SAP$ is such that $\kappa(\phi_l) \cdot \kappa(\phi_r) < 0$ and $\kappa(\phi_l) + \kappa(\phi_r) \neq 0$, the Wiener-Hopf plus Hankel operator $(W+H)_\phi$ and the Wiener-Hopf minus Hankel operator $(W-H)_\phi$ are normally solvable although the Wiener-Hopf operator W_ϕ is not normally solvable. In particular, this exemplifies the differences in the regularity properties of the Wiener-Hopf plus/minus Hankel operators by adding or subtracting the Hankel operator to the Wiener-Hopf operator.

4.1.3 A stronger version of the Sarason's type theorem

Following the main idea of the approach of R. V. Duduchava and A. I. Saginashvili (cf. [29]), we reach a different version of the Sarason's type theorem presented before. Combining then these two versions, we can settle a stronger version of the Sarason's type theorem. Before that, and in order to establish the second version of the Sarason's type theorem, we will first state and prove some lemmas that will be needed in the proof of this result.

Lemma 4.1.4. *Let $\phi, \varphi \in \mathcal{G}SAP$ and suppose that their almost periodic representatives $\phi_l, \varphi_l, \phi_r, \varphi_r$ are connected by*

$$\phi_l = \psi_l^- \varphi_l \psi_l^+, \text{ with } \psi_l^\pm \in \mathcal{G}AP^\pm \text{ and } \mathbf{d}(\psi_l^\pm) = 1, \tag{4.1.13}$$

$$\phi_r = \psi_r^- \varphi_r \psi_r^+, \text{ with } \psi_r^\pm \in \mathcal{G}AP^\pm \text{ and } \mathbf{d}(\psi_r^\pm) = 1. \tag{4.1.14}$$

Then there exist $\zeta_- \in \mathcal{G}(C(\dot{\mathbb{R}}) + H_-^\infty(\mathbb{R})) \cap \mathcal{G}SAP$ and $\zeta_e \in \mathcal{G}L^\infty(\mathbb{R})$, $\widetilde{\zeta_e} = \zeta_e$, such that

$$\phi = \zeta_- \varphi\, \zeta_e\,. \tag{4.1.15}$$

Proof. In [12, Lemma 3.11], it is guaranteed the existence of $\gamma_\pm \in \mathcal{G}(C(\dot{\mathbb{R}}) + H_\pm^\infty(\mathbb{R})) \cap \mathcal{G}SAP$ such that

$$\phi = \gamma_- \varphi\, \gamma_+. \tag{4.1.16}$$

Thus, if we consider

$$\zeta_- = \gamma_- \widetilde{\gamma_+^{-1}}, \tag{4.1.17}$$

$$\zeta_e = \widetilde{\gamma_+} \gamma_+, \tag{4.1.18}$$

we obtain

$$\phi = \zeta_- \varphi\, \zeta_e \tag{4.1.19}$$

with $\zeta_- \in \mathcal{G}(C(\dot{\mathbb{R}}) + H_-^\infty(\mathbb{R})) \cap \mathcal{G}SAP$ and $\zeta_e \in \mathcal{G}L^\infty(\mathbb{R})$, $\widetilde{\zeta_e} = \zeta_e$. ☐

Lemma 4.1.5. *Let $\phi, \varphi \in L^\infty(\mathbb{R})$. If there are functions $\zeta_- \in \mathcal{G}(C(\dot{\mathbb{R}}) + H_-^\infty(\mathbb{R}))$ and $\zeta_e \in \mathcal{G}L^\infty(\mathbb{R})$, $\widetilde{\zeta_e} = \zeta_e$, such that $\phi = \zeta_- \varphi\, \zeta_e$, then the operator $(W+H)_\phi$ $\big($resp. $(W-H)_\phi\big)$*

is properly n-normal, properly d-normal or Fredholm operator if and only if the operator $(W+H)_\varphi$ $\big($resp. $(W-H)_\varphi\big)$ enjoys the same property.

Proof. From the factorization of the Wiener-Hopf plus Hankel operators presented in (1.3.63), we obtain

$$(W+H)_\phi = (W+H)_{\zeta_-\varphi}\,\ell_0\,(W+H)_{\zeta_e}\,. \tag{4.1.20}$$

Applying now Proposition 1.3.2 to $(W+H)_{\zeta_-\varphi}$, it yields

$$(W+H)_{\zeta_-\varphi} = (W+H)_{\zeta_-}\,\ell_0\,(W+H)_\varphi + H_{\zeta_-}\,\ell_0\,(W+H)_{\widetilde{\varphi}-\varphi}\,. \tag{4.1.21}$$

Since $\zeta_- \in \mathcal{G}(C(\dot{\mathbb{R}}) + H^\infty_-(\mathbb{R}))$, in virtue of a theorem due to P. Hartman (see [40], [12, Theorem 2.18]), we have that H_{ζ_-} is a compact operator. Therefore we may rewrite (4.1.21) as follows

$$(W+H)_{\zeta_-\varphi} = (W+H)_{\zeta_-}\,\ell_0\,(W+H)_\varphi + C \tag{4.1.22}$$

with C being the compact operator $H_{\zeta_-}\,\ell_0\,(W+H)_{\widetilde{\varphi}-\varphi}$. Combining (4.1.20) and (4.1.22), we obtain

$$(W+H)_\phi = (W+H)_{\zeta_-}\ell_0\;\;(W+H)_\varphi\;\;\ell_0\,(W+H)_{\zeta_e} + K \tag{4.1.23}$$

where K is the compact operator $H_{\zeta_-}\,\ell_0\,(W+H)_{\widetilde{\varphi}-\varphi}\,\ell_0\,(W+H)_{\zeta_e}$. Now, due to a theorem of R. G. Douglas (cf. [27, 28], [12, Theorem 2.19]), we know that W_{ζ_-} is a Fredholm operator, and consequently, $(W+H)_{\zeta_-}$ is a Fredholm operator (as the sum of a Fredholm Wiener--Hopf operator with a compact Hankel operator). Clearly, since $(W+H)_{\zeta_-}$ is a Fredholm operator, $(W+H)_{\zeta_-}\ell_0$ will be also a Fredholm operator. Finally, from Proposition 1.3.5, it follows that $\ell_0(W+H)_{\zeta_e}$ is a Fredholm operator too. This, in connection with (4.1.23), means that $(W+H)_\phi$ is properly n-normal, properly d-normal or Fredholm operator if and only if $(W+H)_\varphi$ has the same property.

Due to (1.3.76) and (1.3.77), the proof for the Wiener-Hopf minus Hankel case runs identically. $\square$

Proposition 4.1.6. *Let $\phi, \varphi \in \mathcal{G}SAP$ and suppose that their almost periodic representatives $\phi_l, \varphi_l, \phi_r, \varphi_r$ are connected by*

$$\phi_l = \psi_l^- \varphi_l \psi_l^+, \text{ with } \psi_l^\pm \in \mathcal{G}AP^\pm \text{ and } \mathbf{d}(\psi_l^\pm) = 1, \tag{4.1.24}$$

$$\phi_r = \psi_r^- \varphi_r \psi_r^+, \text{ with } \psi_r^\pm \in \mathcal{G}AP^\pm \text{ and } \mathbf{d}(\psi_r^\pm) = 1. \tag{4.1.25}$$

Then the operator $(W+H)_\phi$ $\big($resp. $(W-H)_\phi\big)$ is properly n-normal, properly d-normal or Fredholm if and only if the operator $(W+H)_\varphi$ $\big($resp. $(W-H)_\varphi\big)$ enjoys the same property.

Proof. It runs immediately from Lemmas 4.1.4 and 4.1.5. □

After having proved these auxiliary results, we present next the second version of the Sarason's type theorem. In what follows, let $PC(\mathbb{T})$ denote the C^*-algebra of all *piecewise continuous functions on* $\mathbb{T}$, i.e., functions $\theta \in L^\infty(\mathbb{T})$ for which the one-sided limits

$$\theta(\tau - 0) := \lim_{\varepsilon \to 0^-} \theta(\tau e^{i\varepsilon}), \quad \theta(\tau + 0) := \lim_{\varepsilon \to 0^+} \theta(\tau e^{i\varepsilon}) \tag{4.1.26}$$

exist for each $\tau \in \mathbb{T}$.

Theorem 4.1.7. *Let $\phi \in \mathcal{G}SAP$.*

(a) *If $\kappa(\phi_l) > 0$ and $\kappa(\phi_r) > 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are properly n-normal and left-invertible.*

(b) *If $\kappa(\phi_l) < 0$ and $\kappa(\phi_r) < 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are properly d-normal and right-invertible.*

(c) *Let $\kappa(\phi_l) = \kappa(\phi_r) = 0$.*

   (i) *$(W+H)_\phi$ is a Fredholm operator if and only if $\frac{1}{2\pi} \arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \notin \mathbb{Z} + \frac{1}{4}$.*

   (ii) *$(W-H)_\phi$ is a Fredholm operator if and only if $\frac{1}{2\pi} \arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \notin \mathbb{Z} - \frac{1}{4}$.*

Proof. Applying the same reasoning as in the proof of Theorem 4.1.2, we obtain the following representation of ϕ:

$$\phi = (1-u)\,\mathbf{d}(\phi_l) e_{\kappa(\phi_l)} e^{\omega_l} + u\,\mathbf{d}(\phi_r) e_{\kappa(\phi_r)} e^{\omega_r} + \phi_0, \tag{4.1.27}$$

with $\omega_l, \omega_r \in AP$ and $M(\omega_l) = M(\omega_r) = 0$.

Consider now

$$\varphi = (1-u)\,\mathbf{d}(\phi_l)e_{\kappa(\phi_l)} + u\,\mathbf{d}(\phi_r)e_{\kappa(\phi_r)} + \varphi_0, \tag{4.1.28}$$

where $\varphi_0 \in C_0(\dot{\mathbb{R}})$ is chosen in order to $\varphi \in \mathcal{G}SAP$.

In the first part of this proof, we will prove that $(W+H)_\phi$ is properly n-normal, properly d-normal or Fredholm if and only if $(W+H)_\varphi$ enjoys the same property.

Suppose that $(W+H)_\phi$ is properly n-normal. Then, by a well-known property concerning to the index of semi-Fredholm operators (cf. Theorem 1.2.1), there exists an $\delta > 0$ such that

$$\mathrm{Ind}\,(W+H)_\varrho = \mathrm{Ind}\,(W+H)_\phi, \tag{4.1.29}$$

for all operators $(W+H)_\varrho$ with Fourier symbols ϱ satisfying the condition $\|\phi - \varrho\|_\infty < \delta$. This means that there exists an $\delta > 0$ such that $(W+H)_\varrho$ is properly n-normal for ϱ satisfying $\|\phi - \varrho\|_\infty < \delta$. Let $p_l^\pm, p_r^\pm \in AP^\pm$ be almost periodic polynomials such that

$$M(p_l^\pm) = M(p_l^\pm) = 0, \tag{4.1.30}$$

$$\|\omega_l - p_l^- - p_l^+\|_\infty < \varepsilon, \tag{4.1.31}$$

$$\|\omega_r - p_r^- - p_r^+\|_\infty < \varepsilon, \tag{4.1.32}$$

for $\varepsilon > 0$. Consider now

$$\zeta = (1-u)\,\mathbf{d}(\phi_l)e^{p_l^-}e_{\kappa(\phi_l)}e^{p_l^+} + u\,\mathbf{d}(\phi_r)e^{p_r^-}e_{\kappa(\phi_r)}e^{p_r^+} + \phi_0. \tag{4.1.33}$$

If, in (4.1.31) and (4.1.32), we chose a sufficiently small ε, then we obtain $\|\phi - \zeta\|_\infty < \delta$, which assures that $(W+H)_\zeta$ is properly n-normal. According to Theorem 1.4.1, it holds that $\zeta \in \mathcal{G}L^\infty(\mathbb{R})$. Since $\zeta \in SAP$ and being SAP an inverse closed subalgebra in $L^\infty(\mathbb{R})$, it follows that $\zeta \in \mathcal{G}SAP$. Moreover, due to (4.1.30), we have the guarantee that $\mathbf{d}\big(e^{p_l^\pm}\big) = \mathbf{d}\big(e^{p_r^\pm}\big) = 1$. So, ζ and φ are in the conditions of Proposition 4.1.6, and therefore, since $(W+H)_\zeta$ is properly n-normal, we conclude that $(W+H)_\varphi$ is properly n-normal.

Assume now that $(W+H)_\varphi$ is properly n-normal. Consider

$$\gamma = (1-u)\,\mathbf{d}(\phi_l)e_{\kappa(\phi_l)}e^{\sigma_l} + u\,\mathbf{d}(\phi_r)e_{\kappa(\phi_r)}e^{\sigma_r} + \varphi_0, \tag{4.1.34}$$

with $\sigma_l, \sigma_r \in AP$. By Theorem 1.2.1 there exists an $\delta > 0$ such that $(W+H)_\gamma$ is properly n-normal if in (4.1.34) we choose $\|\sigma_l\|_\infty < \delta$ and $\|\sigma_r\|_\infty < \delta$. Following the same reasoning as before, if $(W+H)_\gamma$ is properly n-normal, then $\gamma \in \mathcal{G}SAP$. Define now

$$\eta = (1-u)\,\mathbf{d}(\phi_l)e^{q_l^-}e_{\kappa(\phi_l)}e^{\sigma_l}e^{q_l^+} + u\,\mathbf{d}(\phi_r)e^{q_r^-}e_{\kappa(\phi_r)}e^{\sigma_r}e^{q_r^+} + \eta_0, \tag{4.1.35}$$

where $q_l^\pm, q_r^\pm \in AP^\pm$ are almost periodic polynomials such that $M(q_l^\pm) = M(q_l^\pm) = 0$ (consequently $\mathbf{d}(e^{q_l^\pm}) = \mathbf{d}(e^{q_r^\pm}) = 1$) and $\eta_0 \in C_0(\dot{\mathbb{R}})$ is so that $\eta \in \mathcal{G}SAP$. According to Proposition 4.1.6, we have that $(W+H)_\eta$ is properly n-normal. If, in (4.1.35), we choose $\eta_0 = \phi_0$ and $q_l^\pm, q_r^\pm$ such that

$$\omega_l = \sigma_l + q_l^- + q_l^+ \text{ and } \omega_r = \sigma_r + q_r^- + q_r^+, \tag{4.1.36}$$

we obtain $\eta = \phi$, and therefore, we get that $(W+H)_\phi$ is properly n-normal (note that the last identities put additional conditions to σ_l and σ_r). At this moment, we proved that $(W+H)_\phi$ is properly n-normal if and only if $(W+H)_\varphi$ is also properly n-normal. Analogously it can be shown that $(W+H)_\phi$ is properly d-normal (resp. Fredholm) if and only if $(W+H)_\varphi$ is properly d-normal (resp. Fredholm). Using the same reasoning, we obtain that $(W-H)_\phi$ is properly n-normal, properly d-normal or Fredholm operator if and only if $(W-H)_\varphi$ enjoys the same property.

Due to this, and in a second part, we will prove the theorem for the Wiener-Hopf-Hankel operators $(W\pm H)_\varphi$.

Suppose that $\kappa(\phi_l) < 0$ and $\kappa(\phi_r) < 0$. Due to a result of Sarason (cf. [66, 67]), it follows $\varphi \in C(\dot{\mathbb{R}}) + H^\infty_-(\mathbb{R})$, and consequently H_φ is a compact operator. From the proof of Theorem 3.9 in [12], we have that W_φ is properly d-normal. Therefore, since $(W\pm H)_\varphi$ is the sum/difference of a properly d-normal Wiener-Hopf operator with a compact Hankel operator, it results that $(W\pm H)_\varphi$ is a properly d-normal operator (cf. [34, Theorem 15.3]), which completes the proof of part (b) of the theorem. Part (a) derives from part (b) by passage to adjoint operators, and in this way we prove that $(W\pm H)_\varphi$ is properly n-normal if $\kappa(\phi_l) > 0$ and $\kappa(\phi_r) > 0$.

Finally, consider $\kappa(\phi_l) = \kappa(\phi_r) = 0$. In this case, $\varphi \in \mathcal{G}C(\overline{\mathbb{R}})$ seeing that

$$\varphi = (1-u)\,\mathbf{d}(\phi_l) + u\,\mathbf{d}(\phi_r) + \varphi_0, \tag{4.1.37}$$

with $u \in C(\overline{\mathbb{R}})$ and $\varphi_0 \in C_0(\dot{\mathbb{R}})$. Thus φ can be viewed as a piecewise continuous function with a single jump at ∞. Using the results of E. L. Basor and T. Ehrhardt on Toeplitz plus Hankel operators with $PC(\mathbb{T})$ symbols, we will see how one can prove part (c). We will start by proving the assertion for the Wiener-Hopf minus Hankel operator $(W-H)_\varphi$ and then we will prove for the case of the Wiener-Hopf plus Hankel operator $(W+H)_\varphi$. Recall from Lemma 1.3.13 that the Wiener-Hopf minus Hankel operator $(W-H)_\varphi$ and the Toeplitz plus Hankel operator $(T+H)_{B_0\varphi}$ are equivalent operators. In [5, Corollary 3.2], we find the following result concerning to the Fredholm property of Toeplitz plus Hankel operators with $PC(\mathbb{T})$ symbols: for $\theta \in PC(\mathbb{T})$, the Toeplitz plus Hankel operator $(T+H)_\theta$ is a Fredholm operator if and only if $\theta(\tau \pm 0) \neq 0$ for each $\tau \in \mathbb{T}$ and

$$\frac{1}{2\pi}\arg\left(\frac{\theta(\tau-0)\theta(\bar{\tau}-0)}{\theta(\tau+0)\theta(\bar{\tau}+0)}\right) \notin \mathbb{Z} + \frac{1}{2} \quad \text{for each } \tau \in \mathbb{T}_+, \tag{4.1.38}$$

$$\frac{1}{2\pi}\arg\left(\frac{\theta(\tau-0)}{\theta(\tau+0)}\right) \notin \mathbb{Z} + \frac{\tau}{4} \quad \text{for each } \tau \in \{-1,1\}. \tag{4.1.39}$$

Here, and in what follows, $\mathbb{T}_+ := \{t \in \mathbb{T} : \Im m\, t > 0\}$. In our case,

$$\begin{aligned}\theta(t) &= B_0\varphi(t)\\ &= \left(1 - u\left(i\frac{1+t}{1-t}\right)\right)\mathbf{d}(\phi_l) + u\left(i\frac{1+t}{1-t}\right)\mathbf{d}(\phi_r) + \varphi_0\left(i\frac{1+t}{1-t}\right),\ t \in \mathbb{T}\backslash\{1\}.\end{aligned} \tag{4.1.40}$$

Since $\varphi \in \mathcal{G}C(\overline{\mathbb{R}})$, it follows that $\theta(\tau \pm 0) \neq 0$ for each $\tau \in \mathbb{T}$. Taking into account that φ is a piecewise continuous function with a single jump at ∞, it yields that $\theta(\tau - 0) = \theta(\tau + 0)$ and $\theta(\bar{\tau} - 0) = \theta(\bar{\tau} + 0)$ for all $\tau \in \mathbb{T}_+$. Consequently, it holds that

$$\frac{1}{2\pi}\arg\left(\frac{\theta(\tau-0)\theta(\bar{\tau}-0)}{\theta(\tau+0)\theta(\bar{\tau}+0)}\right) \in \mathbb{Z} \quad \text{for each } \tau \in \mathbb{T}_+, \tag{4.1.41}$$

which implies that

$$\frac{1}{2\pi}\arg\left(\frac{\theta(\tau-0)\theta(\bar{\tau}-0)}{\theta(\tau+0)\theta(\bar{\tau}+0)}\right) \notin \mathbb{Z} + \frac{1}{2} \quad \text{for each } \tau \in \mathbb{T}_+. \tag{4.1.42}$$

Moreover, due to the equality $\theta(-1-0) = \theta(-1+0)$, it also holds that

$$\frac{1}{2\pi} \arg\left(\frac{\theta(-1-0)}{\theta(-1+0)}\right) \in \mathbb{Z}, \tag{4.1.43}$$

and therefore

$$\frac{1}{2\pi} \arg\left(\frac{\theta(-1-0)}{\theta(-1+0)}\right) \notin \mathbb{Z} - \frac{1}{4}. \tag{4.1.44}$$

Then, since

$$\theta(1-0) = (1-u(+\infty))\,\mathbf{d}(\phi_l) + u(+\infty)\,\mathbf{d}(\phi_r) + \varphi_0(+\infty) = \mathbf{d}(\phi_r), \tag{4.1.45}$$

$$\theta(1+0) = (1-u(-\infty))\,\mathbf{d}(\phi_l) + u(-\infty)\,\mathbf{d}(\phi_r) + \varphi_0(-\infty) = \mathbf{d}(\phi_l), \tag{4.1.46}$$

it follows that

$$\frac{1}{2\pi} \arg\left(\frac{\theta(1-0)}{\theta(1+0)}\right) = \frac{1}{2\pi} \arg\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}\right). \tag{4.1.47}$$

Therefore, by [5, Corollary 3.2] stated above (cf. (4.1.38) and (4.1.39)), we conclude that the Toeplitz plus Hankel operator $(T+H)_{B_0\varphi}$ is a Fredholm operator if and only if

$$\frac{1}{2\pi} \arg\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}\right) \notin \mathbb{Z} + \frac{1}{4}. \tag{4.1.48}$$

In this way, the Wiener-Hopf minus Hankel operator $(W-H)_\varphi$ is also a Fredholm operator (since $(W-H)_\varphi$ and $(T+H)_{B_0\varphi}$ are equivalent operators) if and only if

$$\frac{1}{2\pi} \arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) = -\frac{1}{2\pi} \arg\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}\right) \notin \mathbb{Z} - \frac{1}{4}. \tag{4.1.49}$$

Now that we already proved the Fredholm characterization of the Wiener-Hopf minus Hankel operator $(W-H)_\varphi$, we will prove the Fredholm characterization of the Wiener-Hopf plus Hankel operator $(W+H)_\varphi$. From Lemma 1.3.11, we have that the Wiener-Hopf plus Hankel operator $(W+H)_\varphi$ and the Toeplitz minus Hankel operator $(T-H)_{B_0\varphi}$ are equivalent operators. Using [30, Theorem A.4], we obtain a similar result of [5, Corollary 3.2] for Toeplitz minus Hankel operators. In this case, the Fredholm characterization is the following: for $\theta \in PC(\mathbb{T})$, the Toeplitz minus Hankel operator $(T-H)_\theta$ is a Fredholm operator if and only if $\theta(\tau \pm 0) \neq 0$ for each $\tau \in \mathbb{T}$ and

$$\frac{1}{2\pi} \arg\left(\frac{\theta(\tau-0)\theta(\bar{\tau}-0)}{\theta(\tau+0)\theta(\bar{\tau}+0)}\right) \notin \mathbb{Z} + \frac{1}{2} \quad \text{for each } \tau \in \mathbb{T}_+, \tag{4.1.50}$$

$$\frac{1}{2\pi} \arg\left(\frac{\theta(\tau-0)}{\theta(\tau+0)}\right) \notin \mathbb{Z} - \frac{\tau}{4} \quad \text{for each } \tau \in \{-1, 1\}. \tag{4.1.51}$$

From (4.1.42) and (4.1.43), we already know that

$$\frac{1}{2\pi}\arg\left(\frac{\theta(\tau-0)\theta(\bar{\tau}-0)}{\theta(\tau+0)\theta(\bar{\tau}+0)}\right) \notin \mathbb{Z}+\frac{1}{2} \quad \text{for each } \tau \in \mathbb{T}_+, \tag{4.1.52}$$

$$\frac{1}{2\pi}\arg\left(\frac{\theta(-1-0)}{\theta(-1+0)}\right) \notin \mathbb{Z}+\frac{1}{4}. \tag{4.1.53}$$

Therefore, from (4.1.47), we conclude that the Toeplitz minus Hankel operator $(T-H)_{B_0\varphi}$ is a Fredholm operator if and only if

$$\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}\right) \notin \mathbb{Z}-\frac{1}{4}, \tag{4.1.54}$$

i.e., if and only if

$$\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) = -\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}\right) \notin \mathbb{Z}+\frac{1}{4}. \tag{4.1.55}$$

Finally, due to the equivalence between the Wiener-Hopf plus Hankel operator $(W+H)_\varphi$ and the Toeplitz minus Hankel operator $(T-H)_{B_0\varphi}$, we have that the Wiener-Hopf plus Hankel operator $(W+H)_\varphi$ is a Fredholm operator if and only if (4.1.55) holds true. □

Remark 4.1.8. Looking for both versions of the Sarason's type theorem (cf. Theorems 4.1.2 and 4.1.7), we see that second version of the Sarason's type theorem is more restrictive than the first one in what concerns the assumptions. This occurs since the second version considers only the case where the mean motions $\kappa(\phi_l)$ and $\kappa(\phi_r)$ have the same sign, while the first version considers all the possibilities for the values of $\kappa(\phi_l)$ and $\kappa(\phi_r)$. Consequently, the fact of conditions presented in the second version of the Sarason's type theorem being more refined allow the achievement of a characterization more accurate than the characterization presented in the first version of the Sarason's type theorem.

After having proved the second version of the Sarason's type theorem, we are now in the position to present a stronger version of the Sarason's type theorems provided in Theorems 4.1.2 and 4.1.7.

Theorem 4.1.9. *Let $\phi \in \mathcal{G}SAP$.*

(a) *If $\kappa(\phi_l) + \kappa(\phi_r) < 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are right-invertible, and at least one of these operators is properly d-normal. Moreover, if $\kappa(\phi_l) < 0$ and $\kappa(\phi_r) < 0$, then both operators are properly d-normal.*

(b) *If $\kappa(\phi_l) + \kappa(\phi_r) > 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are left-invertible, and at least one of these operators is properly n-normal. Moreover, if $\kappa(\phi_l) > 0$ and $\kappa(\phi_r) > 0$, then both operators are properly n-normal.*

(c) *If $\kappa(\phi_l) + \kappa(\phi_r) = 0$ and $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \notin \mathbb{Z} \pm \frac{1}{4}$, then $(W+H)_\phi$ and $(W-H)_\phi$ are Fredholm operators.*

(d) *If $\kappa(\phi_l) + \kappa(\phi_r) = 0$ and $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \in \mathbb{Z} + \frac{1}{4}$ or $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \in \mathbb{Z} - \frac{1}{4}$, then at least one of the operators $(W+H)_\phi$ and $(W-H)_\phi$ is not normally solvable.*

(e) *Let $\kappa(\phi_l) = \kappa(\phi_r) = 0$.*

(i) *$(W+H)_\phi$ is a Fredholm operator if and only if $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \notin \mathbb{Z} + \frac{1}{4}$.*

(ii) *$(W-H)_\phi$ is a Fredholm operator if and only if $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \notin \mathbb{Z} - \frac{1}{4}$.*

(iii) *$(W+H)_\phi$ is a Fredholm operator and $(W-H)_\phi$ is not a normally solvable operator if and only if $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \in \mathbb{Z} - \frac{1}{4}$.*

(iv) *$(W-H)_\phi$ is a Fredholm operator and $(W+H)_\phi$ is not a normally solvable operator if and only if $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \in \mathbb{Z} + \frac{1}{4}$.*

Proof. It follows immediately from Theorems 4.1.2 and 4.1.7. Notice that condition

$$\Re e\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \neq 0 \tag{4.1.56}$$

that appears in Theorem 4.1.2 is equivalent to the condition

$$\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \notin \mathbb{Z} \pm \frac{1}{4}. \tag{4.1.57}$$

□

4.1.4 Another look at the invertibility and semi-Fredholm criteria of Wiener-Hopf-Hankel operators with AP symbols

Recall that the invertibility and semi-Fredholm criterion for Wiener-Hopf plus Hankel operators with AP Fourier symbols obtained in Section 3.3 was establish under the assumption on the AP asymmetric factorization of the Fourier symbol of the corresponding operator. There, the value of the index of the middle term of the factorization is what determines if the operator is properly d-normal and right-invertible, properly n-normal and left-invertible or invertible. This occurs contrarily to what happens in the case of Wiener-Hopf plus Hankel operators with APW symbols, where conditions for left, right and both-sided invertibility, and for the semi-Fredholm properties (properly d-normal or properly n-normal) of the operators are obtained from the mean motion of the Fourier symbol of the operator. Using the second version of the Sarason's type theorem for Wiener--Hopf-Hankel operators with SAP symbols (see Theorem 4.1.7), we are able to reformulate the invertibility and semi-Fredholm criterion for Wiener-Hopf-Hankel operators with AP symbols in terms of the value of the mean motion of the Fourier symbol of the operator.

Theorem 4.1.10. *Let $\phi \in \mathcal{G}AP$ admit an AP asymmetric factorization.*

(a) *If $k(\phi) < 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are properly d-normal and right-invertible.*

(b) *If $k(\phi) > 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are properly n-normal and left-invertible.*

(c) *If $k(\phi) = 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are invertible.*

Therefore, if $(W+H)_\phi$ $\big($resp. $(W-H)_\phi\big)$ is a Fredholm operator then $(W+H)_\phi$ $\big($resp. $(W-H)_\phi\big)$ is invertible.

Proof. Suppose that

$$\phi = \phi_- \, e_\lambda \, \phi_e \tag{4.1.58}$$

is an AP asymmetric factorization of ϕ. From Theorem 3.3.1 and Remark 3.3.4, we obtain:

(A) if $\lambda < 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are properly d-normal and right-invertible;

(B) if $\lambda > 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are properly n-normal and left-invertible;

(C) if $\lambda = 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are invertible.

Since $\phi \in \mathcal{G}AP$, then $\phi \in \mathcal{G}SAP$ and therefore we have

$$\phi_l = \phi_r = \phi. \tag{4.1.59}$$

Due to this,

$$\kappa(\phi_l) = \kappa(\phi_r) = \kappa(\phi), \tag{4.1.60}$$

$$\mathbf{d}(\phi_l) = \mathbf{d}(\phi_r) = \mathbf{d}(\phi), \tag{4.1.61}$$

and consequently

$$\kappa(\phi_l) + \kappa(\phi_r) = 2\kappa(\phi), \tag{4.1.62}$$

$$\Re e \left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)} \right) = 1 \neq 0. \tag{4.1.63}$$

Rewriting now Theorem 4.1.7 for $\phi \in \mathcal{G}AP$, one obtains:

(i) if $\kappa(\phi) < 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are properly d-normal and right-invertible;

(ii) if $\kappa(\phi) > 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are properly n-normal and left-invertible;

(iii) if $\kappa(\phi) = 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are Fredholm operators.

From (A)–(C), we observe that the algebra of Wiener-Hopf-Hankel operators with AP Fourier symbols does not contain Fredholm operators that are not invertible. This together with (i)–(iii) completes the proof. □

Remark 4.1.11. Combining Theorems 3.3.1 and 4.1.10, we observe that we can relate the index λ of the AP asymmetric factorization of ϕ with the mean motion of ϕ in the following way:

$$\lambda = C\kappa(\phi),\ C > 0, \tag{4.1.64}$$

i.e.,

$$\operatorname{sgn} \lambda = \operatorname{sgn} \kappa(\phi). \tag{4.1.65}$$

The achievement of an invertibility and semi-Fredholm criterion for Wiener-Hopf-Hankel operators with AP Fourier symbols without the assumption on the AP asymmetric factorization of the Fourier symbol of the operator is also possible and follows from Theorem 4.1.7. In this case, statements (a) and (b) of Theorem 4.1.10 remain true, and the hypothesis $k(\phi) = 0$ gives us the Fredholm property of the Wiener-Hopf-Hankel operators while in Theorem 4.1.10 (with the assumption on the AP asymmetric factorization of the Fourier symbol) we obtain the invertibility of the operators.

Theorem 4.1.12. *Let $\phi \in \mathcal{G}AP$.*

(a) *If $k(\phi) < 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are properly d-normal and right-invertible.*

(b) *If $k(\phi) > 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are properly n-normal and left-invertible.*

(c) *If $k(\phi) = 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are Fredholm operators.*

Proof. It follows directly from Theorem 4.1.7. □

4.2 A Duduchava-Saginashvili's type theorem

In this section, motivated by the Duduchava-Saginashvili's Theorem, we present a Duduchava-Saginashvili's type theorem for the Wiener-Hopf-Hankel operators under study. This result is a generalization of the Sarason's type theorem presented in Theorem 4.1.2 for Wiener-Hopf-Hankel operators acting in L^p Lebesgue spaces $(1 < p < \infty)$.

Since, from now on, we will deal with convolution type operators acting in L^p Lebesgue spaces, and in order to ensure the boundedness of the operators, we will need to assume that the Fourier symbols of the operators belong to the algebra of Fourier multipliers in L^p. For this reason, we will deal with convolution type operators such that its Fourier symbols will belong to a particular class of semi-almost periodic functions. Due to this, we begin this section by presenting the definition of some particular classes of almost and semi-almost periodic functions, the classes AP_p and SAP_p, that were introduced by R. V. Duduchava and A. I. Saginashvili in [29]. Then, we present the Duduchava-Saginashvili's Theorem which is the motivation of this section, and finally, we state and prove our Duduchava--Saginashvili's type theorem.

4.2.1 Some particular classes of almost and semi-almost periodic functions

In Section 2.1 was defined the Banach algebra APW. We recall here that APW is the set of all almost periodic functions which can be written in the form of an absolutely convergent series:

$$\varphi(x) = \sum_j \varphi_j e^{i\lambda_j x} \;\; (x \in \mathbb{R}), \quad \lambda_j \in \mathbb{R}, \quad \varphi_j = M(\varphi\, e_{-\lambda_j}), \quad \sum_j |\varphi_j| < \infty\,. \tag{4.2.66}$$

Since $\mathcal{F}^{-1} e_\lambda \cdot \mathcal{F}$ is a shift operator, it follows that the almost periodic polynomials belong to $\mathcal{M}^p(\mathbb{R})$, and therefore we have the inclusion $APW \subset \mathcal{M}^p(\mathbb{R})$. Due to this inclusion, let AP_p denote the closure of APW in $\mathcal{M}^p(\mathbb{R})$. As already pointed out before, $\mathcal{M}_2 = L^\infty(\mathbb{R})$, and therefore we have that AP_2 coincides with AP. Moreover, we also have

$$APW \subset AP_p \subset AP. \tag{4.2.67}$$

For a function in $\mathcal{G}AP_p$, R. V. Duduchava and A. I. Saginashvili presented a similar result of Bohr's Theorem (see [29]). That is, every function $\phi \in \mathcal{G}AP_p$ has a representation of the form

$$\phi = e_{\kappa(\phi)}\, e^{\psi}\,, \tag{4.2.68}$$

where $\kappa(\phi)$ is a real number and $\psi \in AP_p$.

Recalling now that $C(\dot{\mathbb{R}})$ denotes the space of all (bounded) continuous (complex--valued) functions on $\mathbb{R}$ for which both limits at $\pm\infty$ exist and coincide, and $C(\overline{\mathbb{R}})$ denotes the set of all (bounded) continuous (complex-valued) functions on $\mathbb{R}$ with a possible jump at ∞, let $C_p(\dot{\mathbb{R}})$ and $C_p(\overline{\mathbb{R}})$ denote the closures in $\mathcal{M}^p(\mathbb{R})$ of $C(\dot{\mathbb{R}})$ and $C(\overline{\mathbb{R}})$, respectively.

After we have defined AP_p, $C_p(\dot{\mathbb{R}})$, and $C_p(\overline{\mathbb{R}})$, we are now in position to define the class of Fourier symbols of the Wiener-Hopf-Hankel operators that will be on focus in this section: the algebra SAP_p. Following the spirit of characterization (2.2.27), let SAP_p denote the set of all functions of the form

$$\phi = (1-u)\phi_l + u\phi_r + \phi_0, \tag{4.2.69}$$

where $\phi_l, \phi_r \in AP_p$, $\phi_0 \in C_p(\dot{\mathbb{R}})$ such that $\phi_0(\infty) = 0$ and u is a monotonically increasing real-valued function in $C(\overline{\mathbb{R}})$ satisfying $u(-\infty) = 0$ and $u(+\infty) = 1$. The definition of SAP_p was first introduced by R. V. Duduchava and A. I. Saginashvili in [29].

Similarly to case of the algebra of semi-almost periodic functions, the elements ϕ_l, ϕ_r and ϕ_0 are uniquely determined by ϕ, and ϕ_l, ϕ_r are independent of the choice of u, and the maps which perform the transformations

$$\phi \mapsto \phi_l, \quad \phi \mapsto \phi_r \tag{4.2.70}$$

are Banach algebra homomorphisms of SAP_p onto AP_p.

From the definition of SAP_p, it follows that $SAP_p \subset \mathcal{M}^p(\mathbb{R})$ and that SAP_p contains the algebras AP_p and $C_p(\overline{\mathbb{R}})$. In fact, SAP_p is the smallest closed subalgebra of $\mathcal{M}^p(\mathbb{R})$ that contains AP_p and $C_p(\overline{\mathbb{R}})$. This is another possibility for defining SAP_p (which mimics the definition of SAP) that was presented by Yu. I. Karlovich and I. M. Spitkovsky. We can find this definition in [44], where it was also proved that both definitions lead to the same class of functions (cf. [44, Lemma 3.1]).

Finally, please notice that $SAP_2 = SAP$ and, consequently, we have

$$SAP_p \subset SAP. \tag{4.2.71}$$

4.2.2 The Duduchava-Saginashvili's theorem

For all $\psi \in PC$ (in particular, for all $\psi \in C(\overline{\mathbb{R}})$), the function $\psi^\# : \dot{\mathbb{R}} \times [0,1] \to \mathbb{C}$ is defined by

$$\psi^\#(x,\mu) := (1-\mu)\,\psi(x-0) + \mu\,\psi(x+0). \tag{4.2.72}$$

The range of $\psi^\#$ is a continuous closed curve with a natural orientation induced by the orientation of $\mathbb{R}$ from $-\infty$ to $+\infty$. Basically, $\psi^\#$ is obtained from ψ by joining, in each jump of ψ, the points $\psi(x_0 - 0)$ and $\psi(x_0 + 0)$ with a line segment. In this way, the line segment $[\psi(x_0 - 0), \psi(x_0 + 0)]$ is oriented from $\psi(x_0 - 0)$ to $\psi(x_0 + 0)$.

For Wiener-Hopf operators with PC Fourier symbols and acting in L^2 Lebesgue spaces, there is a result that asserts the Fredholm property of the Wiener-Hopf operators based on $\psi^{\#}$. The result is the following: considering $\psi \in PC\backslash\{0\}$, if $0 \in \operatorname{Im}\psi^{\#}$, then W_ψ is not normally solvable; otherwise, if $0 \notin \operatorname{Im}\psi^{\#}$, then W_ψ is a Fredholm operator. Clearly, this result states the fundamental role of condition $0 \in \operatorname{Im}\psi^{\#}$ in order to decide if the Wiener-Hopf operator is a Fredholm operator or if it is not a normally solvable operator.

Considering now Wiener-Hopf operators with PC_p Fourier symbols and acting in L^p Lebesgue spaces, there is also a well-known result concerning to the Fredholm property of the operators and based on continuous functions which depend on the Fourier symbols of the operators. To establish this result, we need first to define these continuous functions dependent on the Fourier symbols of the operators.

Consider the function $\sigma_p : \overline{\mathbb{R}} \to \mathbb{C}$ given by

$$\sigma_p(\mu) := \frac{1}{2} + \frac{1}{2}\coth\left[\pi\left(\frac{i}{p} + \mu\right)\right] \tag{4.2.73}$$

(where the number $p \in (1, \infty)$ is the Lebesgue index of the spaces). Since $\sigma_p(-\infty) = 0$ and $\sigma_p(+\infty) = 1$, $\sigma_p(\mu)$ runs along the circular arc joining 0 and 1, whenever μ runs from $-\infty$ to $+\infty$. Moreover, from the points of the arc σ_p, the line segment $[0, 1]$ is seen at the angle $2\pi/\max\{p, q\}$ $(1/p + 1/q = 1)$, and $\sigma_p(\mu)$ lies on the left (resp. right) of the line segment $[0, 1]$ if $1 < p < 2$ (resp. $2 < p < \infty$). In the particular case of $p = 2$, the arc σ_p coincides with the line segment $[0, 1]$.

Given two points $z_1, z_2 \in \mathbb{C}$, we will denote by $\mathcal{A}_p(z_1, z_2)$ the circular arc that joints the points z_1 and z_2 as follows

$$\mathcal{A}_p(z_1, z_2) := \left\{z_1 + (z_2 - z_1)\,\sigma_p(\mu), \mu \in \overline{\mathbb{R}}\right\}. \tag{4.2.74}$$

Figure 4.1 illustrates the definition of $\mathcal{A}_p(z_1, z_2)$, where are represented the circular arcs $\mathcal{A}_p(z_1, z_2)$ for several values of p.

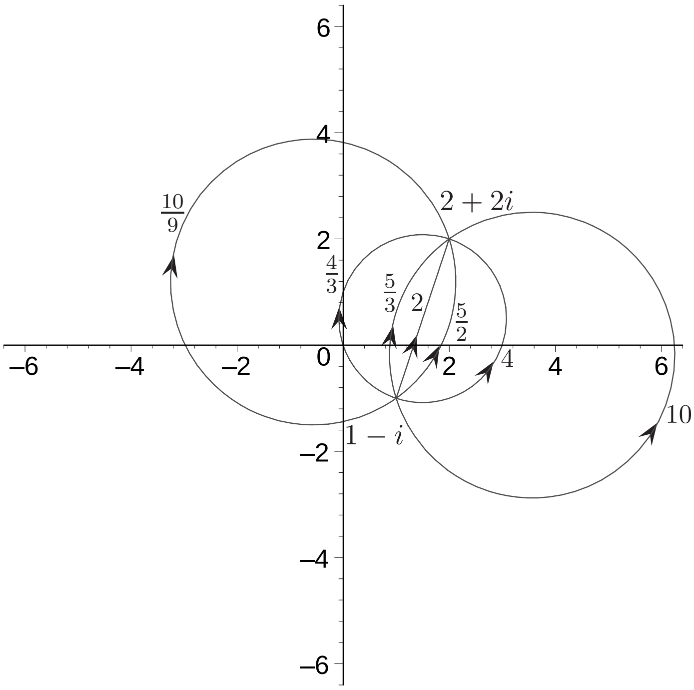


Figure 4.1: The circular arcs $\mathcal{A}_p\,(1-i, 2+2i)$ for several values of p.

Given now $\psi \in PC$, we define the function $\psi^p : \dot{\mathbb{R}} \times \overline{\mathbb{R}} \to \mathbb{C}$ by

$$\psi^p(x,\mu) := (1-\sigma_p(\mu))\ \psi(x-0) + \sigma_p(\mu)\ \psi(x+0). \tag{4.2.75}$$

The range of ψ^p is a continuous closed curve with a natural orientation induced by the orientation of $\mathbb{R}$ from $-\infty$ to $+\infty$ and it is obtained from the range of ψ by joining, in each jump of ψ, the points $\psi(x_0-0)$ and $\psi(x_0+0)$ with the arc $\mathcal{A}_q\,(\psi(x_0-0), \psi(x_0+0))$ if $x_0 \in \mathbb{R}$, and the points $\psi(+\infty)$ and $\psi(-\infty)$ with the arc $\mathcal{A}_p\,(\psi(+\infty), \psi(-\infty))$ if ψ has a jump at ∞.

After defining the function ψ^p, we can now present a Fredholm criterion for Wiener--Hopf operators with PC_p Fourier symbols and acting in L^p spaces. That is, for $\psi \in PC_p$, the Wiener-Hopf operator W_ψ is Fredholm if and only if $\psi^p(x,\mu) \neq 0$ for all $(x,\mu) \in \dot{\mathbb{R}} \times \overline{\mathbb{R}}$ (cf. e.g. [14]). Similarly to the Fredholm criterion for Wiener-Hopf operators with PC Fourier symbols and acting in L^2 Lebesgue spaces, the condition $0 \notin \operatorname{Im} \psi^p$ is fundamental in order to decide if the Wiener-Hopf operator with PC_p Fourier symbols, acting between

L^p Lebesgue spaces, is a Fredholm operator or not.

For Wiener-Hopf operators with SAP_p Fourier symbols, there is also a result concerning the Fredholm property where the arc of the form $\mathcal{A}_p(z_1, z_2)$ is important to decide if the operator is or is not a Fredholm operator. This result is due to R. V. Duduchava and A. I. Saginashvili and states the following:

Theorem 4.2.1. (cf. [29, Theorems 2.1 and 2.2]) *If* $\phi \in SAP_p \backslash \{0\}$ *and* $1 < p < \infty$, *the Wiener-Hopf operator*

$$W_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+) \tag{4.2.76}$$

is normally solvable if and only if:

(I) $\inf_{x \in \mathbb{R}} |\phi(x)| > 0$;

(II) $\kappa(\phi_l)\kappa(\phi_r) \geq 0$, *and* $\inf_{x\in\mathbb{R}} |\widetilde{\phi}_p(\infty, x)| > 0$ *if* $\kappa(\phi_l) = \kappa(\phi_r) = 0$, *where*

$$\widetilde{\phi}_p(\infty, x) := \frac{1}{2}\big(\mathbf{d}(\phi_r) + \mathbf{d}(\phi_l)\big) - \frac{1}{2}(\mathbf{d}(\phi_r) - \mathbf{d}(\phi_l)) \coth\left[\pi\left(\frac{i}{p} + x\right)\right], x \in \mathbb{R}. \tag{4.2.77}$$

Moreover,

(1) *if* $\kappa(\phi_l) = \kappa(\phi_r) = 0$, *then the operator* $W_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$ *is Fredholm;*

(2) *if* $\kappa(\phi_l)\kappa(\phi_r) \geq 0$ *and* $\kappa(\phi_l) + \kappa(\phi_r) > 0$, *then the operator* $W_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$ *is left-invertible and* $\dim Coker W_\phi = \infty$;

(3) *if* $\kappa(\phi_l)\kappa(\phi_r) \geq 0$ *and* $\kappa(\phi_l) + \kappa(\phi_r) < 0$, *then the operator* $W_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$ *is right-invertible and* $\dim Ker W_\phi = \infty$.

Please notice that, for all $x \in \mathbb{R}$,

$$\widetilde{\phi}_p(\infty, x) = \mathbf{d}(\phi_r) + (\mathbf{d}(\phi_l) - \mathbf{d}(\phi_r))\, \sigma_p(x). \tag{4.2.78}$$

Therefore, condition $\inf_{x\in\mathbb{R}} |\widetilde{\phi}_p(\infty, x)| > 0$ is equivalent to condition $0 \notin \mathcal{A}_p(\mathbf{d}(\phi_r), \mathbf{d}(\phi_l))$. In [12], we find the following version of the Duduchava-Saginashvili's theorem stated below. Although it states the same as Theorem 4.2.1, it turns to be more easy to identify the conditions on the Fourier symbol ϕ (namely, conditions about the arc $\mathcal{A}_p(\mathbf{d}(\phi_r), \mathbf{d}(\phi_l))$) in order to classify the (semi-)Fredholm property of the Wiener-Hopf operator W_ϕ.

Theorem 4.2.2. (cf. [12, Theorem 19.15]) *Let $\phi \in SAP_p \backslash \{0\}$ and $1 < p < \infty$.*

(a) *If $\phi \notin \mathcal{G}SAP_p$, then $W_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$ is not normally solvable.*

(b) *If $\phi \in \mathcal{G}SAP_p$ and $\kappa(\phi_l)\kappa(\phi_r) < 0$, then $W_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$ is not normally solvable.*

(c) *If $\phi \in \mathcal{G}SAP_p$, $\kappa(\phi_l)\kappa(\phi_r) \geq 0$, and $\kappa(\phi_l) + \kappa(\phi_r) > 0$, then $W_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$ is properly n-normal and has trivial kernel on $L^p_+(\mathbb{R})$.*

(d) *If $\phi \in \mathcal{G}SAP_p$, $\kappa(\phi_l)\kappa(\phi_r) \geq 0$, and $\kappa(\phi_l) + \kappa(\phi_r) < 0$, then $W_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$ is properly d-normal and has dense range on $L^p(\mathbb{R}_+)$.*

(e) *If $\phi \in \mathcal{G}SAP_p$, $\kappa(\phi_l) = \kappa(\phi_r) = 0$, and $0 \notin \mathcal{A}_p\left(\mathbf{d}(\phi_r), \mathbf{d}(\phi_l)\right)$, then $W_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$ is a Fredholm operator.*

(f) *If $\phi \in \mathcal{G}SAP_p$, $\kappa(\phi_l) = \kappa(\phi_r) = 0$, and $0 \in \mathcal{A}_p\left(\mathbf{d}(\phi_r), \mathbf{d}(\phi_l)\right)$, then $W_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$ is not normally solvable.*

4.2.3 Main result

Now that we have presented the definitions of SAP_p functions and of the arc $\mathcal{A}_p\left(z_1, z_2\right)$, as well as the motivation of this section - the Duduchava-Saginashvili's theorem - we are now in position to present a Duduchava-Saginashvili's type theorem for Wiener-Hopf-Hankel operators. As we will see, to establish this result we follow the same approach as the one used in the proof of the Sarason's type theorem presented in Theorem 4.1.2, i.e., we use the Δ–relation after extension between the Wiener-Hopf plus Hankel operator and the Wiener--Hopf operator. By studying, in a first place, the regularity properties of the Wiener-Hopf operator (with the help of the Duduchava-Saginashvili's Theorem), we then transfer these regularity properties to the Wiener-Hopf-Hankel operators.

Theorem 4.2.3. *Let $\phi \in \mathcal{G}SAP_p$ and consider $(W \pm H)_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$, with $1 < p < \infty$.*

(a) *If $\kappa(\phi_l) + \kappa(\phi_r) < 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are right-invertible. Moreover, at least one of these operators is properly d-normal.*

(b) *If $\kappa(\phi_l) + \kappa(\phi_r) > 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are left-invertible. In addition, at least one of these operators is properly n-normal.*

(c) *If $\kappa(\phi_l) + \kappa(\phi_r) = 0$ and*

$$0 \notin \mathcal{A}_p\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}, \frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right), \tag{4.2.79}$$

then $(W+H)_\phi$ and $(W-H)_\phi$ are Fredholm operators.

(d) *If $\kappa(\phi_l) + \kappa(\phi_r) = 0$ and*

$$0 \in \mathcal{A}_p\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}, \frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right), \tag{4.2.80}$$

then at least one of the operators $(W+H)_\phi$ and $(W-H)_\phi$ is not normally solvable.

Proof. From the definition of SAP_p, we have the following representation of ϕ,

$$\phi = (1-u)\phi_l + u\phi_r + \phi_0, \tag{4.2.81}$$

where $\phi_l, \phi_r \in AP_p$, $\phi_0 \in C_p(\dot{\mathbb{R}})$ is such that $\phi_0(\infty) = 0$, and u is a monotonically increasing real-valued function in $C(\overline{\mathbb{R}})$ satisfying $u(-\infty) = 0$ and $u(+\infty) = 1$. Applying the same reasoning as in the proof of Theorem 4.1.2, we obtain the following representation of ϕ:

$$\phi = (1-u)\,\mathbf{d}(\phi_l) e_{\kappa(\phi_l)} e^{\omega_l} + u\,\mathbf{d}(\phi_r) e_{\kappa(\phi_r)} e^{\omega_r} + \phi_0, \tag{4.2.82}$$

with $\omega_l, \omega_r \in AP_p$ and $M(\omega_l) = M(\omega_r) = 0$.

Due to the transfer of regularity properties from the Wiener-Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ to the Wiener-Hopf plus Hankel operator $(W+H)_\phi$ and to the Wiener-Hopf minus Hankel operator $(W-H)_\phi$ (cf. Corollary 1.3.8 and Corollary 1.3.10), we will now study the regularity properties of the Wiener-Hopf operator $W_{\phi\widetilde{\phi^{-1}}} : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$. Once again, following the same reasoning as in the proof of Theorem 4.1.2, we obtain

$$\phi\widetilde{\phi^{-1}} = \frac{(1-u)\,\mathbf{d}(\phi_l) e_{\kappa(\phi_l)} e^{\omega_l} + u\,\mathbf{d}(\phi_r) e_{\kappa(\phi_r)} e^{\omega_r} + \phi_0}{(1-\widetilde{u})\,\mathbf{d}(\phi_l) e_{-\kappa(\phi_l)} e^{\widetilde{\omega_l}} + \widetilde{u}\,\mathbf{d}(\phi_r) e_{-\kappa(\phi_r)} e^{\widetilde{\omega_r}} + \widetilde{\phi_0}}, \tag{4.2.83}$$

being the almost periodic representatives of $\widetilde{\phi\phi^{-1}}$ given by

$$\big(\widetilde{\phi\phi^{-1}}\big)_l = \frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\, e_{\kappa(\phi_l)+\kappa(\phi_r)}\, e^{\omega_l-\widetilde{\omega_r}}, \tag{4.2.84}$$

$$\big(\widetilde{\phi\phi^{-1}}\big)_r = \frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}\, e_{\kappa(\phi_l)+\kappa(\phi_r)}\, e^{\omega_r-\widetilde{\omega_l}}. \tag{4.2.85}$$

From this, taking into account that $M(\omega_l) = M(\omega_r) = 0$, we obtain:

$$\kappa\Big(\big(\widetilde{\phi\phi^{-1}}\big)_l\Big) = \kappa\Big(\big(\widetilde{\phi\phi^{-1}}\big)_r\Big) = \kappa(\phi_l) + \kappa(\phi_r), \tag{4.2.86}$$

$$\mathbf{d}\Big(\big(\widetilde{\phi\phi^{-1}}\big)_l\Big) = \frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)},\ \mathbf{d}\Big(\big(\widetilde{\phi\phi^{-1}}\big)_r\Big) = \frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}. \tag{4.2.87}$$

Applying now the Duduchava-Saginashvili's Theorem to the Wiener-Hopf operator $W_{\widetilde{\phi\phi^{-1}}}$ (see Theorem 4.2.2), it follows that:

(a) if $\kappa(\phi_l) + \kappa(\phi_r) < 0$, then $W_{\widetilde{\phi\phi^{-1}}}$ is properly d-normal and right-invertible;

(b) if $\kappa(\phi_l) + \kappa(\phi_r) > 0$, then $W_{\widetilde{\phi\phi^{-1}}}$ is properly n-normal and left-invertible;

(c) if $\kappa(\phi_l) + \kappa(\phi_r) = 0$ and

$$0 \notin \mathcal{A}_p\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}, \frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right), \tag{4.2.88}$$

then $W_{\widetilde{\phi\phi^{-1}}}$ is a Fredholm operator;

(d) if $\kappa(\phi_l) + \kappa(\phi_r) = 0$ and

$$0 \in \mathcal{A}_p\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}, \frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right), \tag{4.2.89}$$

then $W_{\widetilde{\phi\phi^{-1}}}$ is not normally solvable.

Applying Corollary 1.3.8 and Corollary 1.3.10, we obtain that $(W+H)_\phi$ and $(W-H)_\phi$ are right-invertible, left-invertible, and Fredholm operators, under the conditions of case (a), (b) and (c), respectively. Finally, to arrive at the statement, we interpret the Δ–relation after extension between the Wiener-Hopf plus Hankel operator $(W+H)_\phi$ and the Wiener--Hopf operator $W_{\widetilde{\phi\phi^{-1}}}$ (presented in Lemma 1.3.7) as an equivalence after extension between $\mathrm{diag}[(W+H)_\phi, \mathcal{T}_\phi]$ and $W_{\widetilde{\phi\phi^{-1}}}$, and then use the equivalence after extension between the operators $\mathcal{T}_\phi$ and $(W-H)_\phi$ (cf. Proposition 1.3.9). $\square$

Remark 4.2.4. Being the Duduchava-Saginashvili's type theorem a generalization of the Sarason's type theorem presented in Theorem 4.1.2, analogous observations as in the case of Remark 4.1.3 hold true. First of all, Theorem 4.2.3 is called a Duduchava-Saginashvili's type theorem for Wiener-Hopf-Hankel operators since it describes the Fredholm nature of $(W+H)_\phi$ and $(W-H)_\phi$ based on the values of $\kappa(\phi_l)$, $\kappa(\phi_r)$, $\mathbf{d}(\phi_l)$ and $\mathbf{d}(\phi_r)$ when $\phi \in \mathcal{G}SAP_p$. Moreover, Theorem 4.2.3 is not a complete characterization of all the regularity properties of $(W \pm H)_\phi$, since, under the conditions of the theorem, there are still some questions that remain to be answered (namely, in the case of condition (d) concerning the normal solvability of the operators). Finally, comparing Theorem 4.2.3 with the Duduchava-Saginashvili's Theorem, it is easy to see the differences in the regularity properties of the Wiener-Hopf plus/minus Hankel operators by adding or subtracting the Hankel operator to the Wiener-Hopf operator. For instance, for $\phi \in \mathcal{G}SAP_p$ such that $\kappa(\phi_l) \cdot \kappa(\phi_r) < 0$ and $\kappa(\phi_l) + \kappa(\phi_r) \neq 0$, the Wiener-Hopf-Hankel operators $(W \pm H)_\phi$ are normally solvable although the Wiener-Hopf operator W_ϕ is not normally solvable.

4.3 Fredholm index formula

In this section, we establish a Fredholm index formula, in a first place, for Fredholm Wiener-Hopf-Hankel operators with SAP Fourier symbols, acting between L^2 Lebesgue spaces, and then for Fredholm Wiener-Hopf-Hankel operators with SAP_p Fourier symbols, acting between L^p Lebesgue spaces. Since these Fredholm index formulae are obtained in terms of the winding number, we begin this section by recalling the definitions of Cauchy index and winding number of continuous functions.

Consider $\varrho \in \mathcal{G}C(\overline{\mathbb{R}})$. If $\arg \varrho$ is a continuous argument of ϱ, then the limits $(\arg \varrho)(\pm\infty)$ exist and are finite. In this way, the *Cauchy index* of ϱ is defined by

$$\operatorname{ind} \varrho := \frac{1}{2\pi}\big((\arg \varrho)(+\infty) - (\arg \varrho)(-\infty)\big). \tag{4.3.90}$$

It follows from the definition of Cauchy index that $\operatorname{ind} \varrho$ is a real number and is independent of the particular choice of $\arg \phi$. Consider now $\varrho \in \mathcal{G}C(\dot{\mathbb{R}})$. Since, in this case, the curve traced out by the point $\varrho(x)$, as x moves from $-\infty$ to $+\infty$, is a closed continuous oriented curve on $\mathbb{C}\backslash\{0\}$, it holds that $\operatorname{ind}\varrho$ is an integer. Thus, the number of times that the curve traced out by the point $\varrho(x)$, as x moves from $-\infty$ to $+\infty$, surrounds the origin counter-clockwise is defined as the *winding number* of ϱ and it is denoted by $\operatorname{wind}\varrho$. In the particular case of $\varrho \in \mathcal{G}C(\dot{\mathbb{R}})$, one gets $\operatorname{wind}\varrho = \operatorname{ind}\varrho$.

For Wiener-Hopf operators with Fourier symbols in $C(\dot{\mathbb{R}})$, acting between L^2 Lebesgue spaces, there is a result due to I. Gohberg that settles the Fredholm property. Moreover, it also presents a relation between the Fredholm index of the Wiener-Hopf operator and the winding number of its Fourier symbol. The result is the following:

Theorem 4.3.1. ([12, Theorem 2.15], [36]) *If $\varrho \in C(\dot{\mathbb{R}})\backslash\{0\}$ has a zero on $\dot{\mathbb{R}}$, then $W_\phi : L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+)$ is not normally solvable. If $\varrho \in \mathcal{G}C(\dot{\mathbb{R}})$, then $W_\phi : L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+)$ is a Fredholm operator and*

$$\operatorname{Ind} W_\varrho = -\operatorname{wind}\varrho. \tag{4.3.91}$$

As we will see ahead, for Fredholm Wiener-Hopf-Hankel operators with SAP Fourier symbols, acting between L^2 Lebesgue spaces, we establish a similar result for the Fredholm index, i.e., the Fredholm index of Wiener-Hopf-Hankel operators is also given in terms of the winding number of the Fourier symbols of the operators. In the case of Fredholm Wiener-Hopf-Hankel operators with SAP_p Fourier symbols, acting between L^p Lebesgue spaces, we also obtain the Fredholm index given in terms of the winding number of a function, being in this case a continuous function constructed from the Fourier symbol of the operator (instead of the winding number of the Fourier symbol of the operator).

4.3.1 Fredholm index formula for Wiener-Hopf-Hankel operators with SAP Fourier symbols

In [67], D. Sarason presented a formula for the Fredholm index of Toeplitz operators in the Hardy space H^2 (with SAP Fourier symbols) in terms of the winding number of the Fourier symbol. A similar result also holds true for Wiener-Hopf operators $W_\phi : L^2_+(\mathbb{R}) \to$

$L^2(\mathbb{R}_+)$ with Fourier symbols in the algebra of semi-almost periodic functions (see [12, Theorem 3.14]).

According to Sarason's Theorem (cf. Theorem 4.1.1), the Wiener-Hopf operator W_ϕ has the Fredholm property if and only if $\phi \in \mathcal{G}SAP$ is such that $\kappa(\phi_l) = \kappa(\phi_r) = 0$ and $0 \notin [\mathbf{d}(\phi_l), \mathbf{d}(\phi_r)]$. Therefore, in order to present an index formula for Fredholm Wiener--Hopf operators, we just need to consider Wiener-Hopf operators with semi-almost periodic Fourier symbols that satisfy the conditions mentioned above. In this way, to compute the Fredholm index of W_ϕ upon the winding number of ϕ, it follows that the winding number only needs to be defined for $\phi \in \mathcal{G}SAP$ such that W_ϕ is a Fredholm operator.

Before presenting the definition of winding number, we will first introduce the definition of the Cauchy index of a semi-almost periodic function. For that purpose, consider $A \subset \mathbb{R}_+$ to be an unbounded set and $\{I_\alpha\}_{\alpha\in A} = \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ a family of intervals $I_\alpha \subset \mathbb{R}$ such that $|I_\alpha| = y_\alpha - x_\alpha \to \infty$, as $\alpha \to \infty$. In [12, Lemma 3.12], it was proved that if $\phi \in \mathcal{G}SAP$ is such that $\kappa(\phi_l) = \kappa(\phi_r) = 0$, then the limit

$$\Upsilon := \frac{1}{2\pi} \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} \big((\arg\phi)(x) - (\arg\phi)(-x)\big)dx \tag{4.3.92}$$

exists, is finite, and is independent of the particular choice of the family $\{I_\alpha\}$ and the continuous argument of ϕ, $\arg\phi$. In this way, for $\phi \in \mathcal{G}SAP$ such that $\kappa(\phi_l) = \kappa(\phi_r) = 0$, the value in (4.3.92) defines the *Cauchy index* of ϕ and it is denoted by $\operatorname{ind}\phi$.

Concerned with the winding number notion, if $\phi \in \mathcal{G}SAP$ such that $\kappa(\phi_l) = \kappa(\phi_r) = 0$ and $0 \notin [\mathbf{d}(\phi_l), \mathbf{d}(\phi_r)]$, the *winding number* of ϕ is defined by

$$\begin{aligned} \operatorname{wind}\phi &:= \operatorname{ind}\phi - \frac{1}{2} + \left\{\frac{1}{2} + \frac{1}{2\pi}\arg\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right\} \\ &= \operatorname{ind}\phi + \frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right\} \\ &= \operatorname{ind}\phi + \frac{1}{2\pi}\arg\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}, \end{aligned} \tag{4.3.93}$$

where the last equality is valid if and only if $\arg(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) \in (-\pi, \pi)$. Here $\{x\}$ denotes the fractional part $\mu \in [0, 1)$ of the real number $x = n + \mu$, with $n \in \mathbb{Z}$. Now that the definition of winding number for semi-almost periodic functions was introduced, we present the analogue of the result of D. Sarason concerned with the Fredholm index formula of Wiener-Hopf operators with SAP Fourier symbols.

Theorem 4.3.2. (cf. [12, Theorem 3.14] and the proof of [67, Theorem 1]) *If* $\phi \in \mathcal{G}SAP$, $\kappa(\phi_l) = \kappa(\phi_r) = 0$ *and* $0 \notin [\mathbf{d}(\phi_l), \mathbf{d}(\phi_r)]$, *then*

$$\operatorname{Ind} W_\phi = -\operatorname{wind} \phi. \tag{4.3.94}$$

As we will see ahead, a similar result holds true for Wiener-Hopf-Hankel operators. In order to achieve that result, we will start by generalizing the notion of Cauchy index and winding number for semi-almost periodic functions in the framework of Fredholm Wiener-Hopf-Hankel operators.

From the Sarason's type theorem presented in Theorem 4.1.9, we know that $(W+H)_\phi$ and $(W-H)_\phi$ are Fredholm operators if $\kappa(\phi_l)+\kappa(\phi_r) = 0$ and $\frac{1}{2\pi}\arg(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) \notin \mathbb{Z}\pm\frac{1}{4}$. From Theorem 4.1.9, we also know that, in the case where $\kappa(\phi_l) = \kappa(\phi_r) = 0$, $(W+H)_\phi$ is a Fredholm operator if and only if $\frac{1}{2\pi}\arg(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) \notin \mathbb{Z}+\frac{1}{4}$, and $(W-H)_\phi$ is a Fredholm operator if and only if $\frac{1}{2\pi}\arg(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) \notin \mathbb{Z}-\frac{1}{4}$. Thus, in order to present a Fredholm index formula, we will only consider the case of Fredholm Wiener-Hopf-Hankel operators with semi-almost periodic Fourier symbols, i.e., Wiener-Hopf-Hankel operators with semi-almost periodic Fourier symbols $\phi \in \mathcal{G}SAP$ such that ϕ satisfies the conditions mentioned above.

Since the Cauchy index of a semi-almost periodic was defined before for $\phi \in \mathcal{G}SAP$ such that $\kappa(\phi_l) = \kappa(\phi_r) = 0$, let us now see if we are able to generalize this definition for $\phi \in \mathcal{G}SAP$ satisfying the condition $\kappa(\phi_l) + \kappa(\phi_r) = 0$. In this sense, consider $\phi \in \mathcal{G}SAP$ such that $\kappa(\phi_l) + \kappa(\phi_r) = 0$. Due to (2.2.27), it follows that

$$(\arg\phi)(x) = (\arg\phi_r)(x) + o(1), \quad x \to +\infty, \tag{4.3.95}$$

$$(\arg\phi)(-x) = (\arg\phi_l)(-x) + o(1), \quad x \to +\infty. \tag{4.3.96}$$

Recall that $\phi_l, \phi_r \in \mathcal{G}SAP$ because $\phi \in \mathcal{G}SAP$. Therefore, by Bohr's Theorem (cf.

(2.1.13)), we know that we may represent the almost periodic representatives of ϕ as

$$\phi_l = e_{\kappa(\phi_l)}\, e^{\varphi_l}, \tag{4.3.97}$$

$$\phi_r = e_{\kappa(\phi_r)}\, e^{\varphi_r}, \tag{4.3.98}$$

where $\varphi_l, \varphi_r \in AP$. In this case, we have that

$$(\arg \phi_l)(x) = \kappa(\phi_l)x + \Im m\, \varphi_l(x), \tag{4.3.99}$$

$$(\arg \phi_r)(x) = \kappa(\phi_r)x + \Im m\, \varphi_r(x). \tag{4.3.100}$$

Using the last information in the right hand-side of (4.3.92), we get

$$\begin{aligned}
\Upsilon =& \frac{1}{2\pi} \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} \big((\arg \phi_r)(x) - (\arg \phi_l)(-x)\big) dx \\
=& \frac{1}{2\pi} \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} \big(\kappa(\phi_r)x + \Im m\, \varphi_r(x) + \kappa(\phi_l)x - \Im m\, \varphi_l(-x)\big) dx \\
=& \frac{1}{2\pi} \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} \big(\Im m\, \varphi_r(x) - \Im m\, \varphi_l(-x)\big) dx.
\end{aligned} \tag{4.3.101}$$

Since $\varphi_l, \varphi_r \in AP$, we also have $\Im m\, \varphi_l, \Im m\, \varphi_r \in AP$. So, from the definition of the Bohr mean value, it holds that $M(\Im m\, \varphi_l)$ and $M(\Im m\, \varphi_r)$ exist, are finite and independent of the particular choice of the family $\{I_\alpha\}$. Thus, it results that

$$\Upsilon = \frac{1}{2\pi}\Big(M(\Im m\, \varphi_r) - M(\Im m\, \varphi_l)\Big). \tag{4.3.102}$$

In this way, we obtain that, for $\phi \in \mathcal{G}SAP$ with $\kappa(\phi_l) + \kappa(\phi_r) = 0$ (and considering $\arg \phi$ to be any continuous argument of ϕ), the limit in (4.3.92) also exists, is finite, and is independent of the particular choice of $\arg \phi$ and of the family $\{I_\alpha\}$. Therefore, the Cauchy index for $\phi \in \mathcal{G}SAP$ such that $\kappa(\phi_l) + \kappa(\phi_r) = 0$ is well defined if we consider its value Υ given by (4.3.92).

To establish a Fredholm index formula for Wiener-Hopf-Hankel operators in terms of the winding number of ϕ, we need to introduce a precise definition of the winding number of ϕ in the case where $\phi \in \mathcal{G}SAP$ is such that $\kappa(\phi_l) + \kappa(\phi_r) = 0$, $\Re e(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) \neq 0$ and $0 \notin [\mathbf{d}(\phi_l), \mathbf{d}(\phi_r)]$. Notice that the case of $\phi \in \mathcal{G}SAP$ such that $\kappa(\phi_l) = \kappa(\phi_r) = 0$, $0 \notin [\mathbf{d}(\phi_l), \mathbf{d}(\phi_r)]$ and $\frac{1}{2\pi} \arg\left(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)\right) \notin \mathbb{Z}+\frac{1}{4}$ or $\frac{1}{2\pi} \arg\left(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)\right) \notin \mathbb{Z}-\frac{1}{4}$ is already considered in the definition of winding number of semi-almost periodic functions presented before in (4.3.93). The generalization of the definitions of Cauchy index and winding number in the framework of Fredholm Wiener-Hopf-Hankel operators is the following.

Definition 4.3.3. *For $\phi \in \mathcal{G}SAP$ such that $\kappa(\phi_l) + \kappa(\phi_r) = 0$, we define the value Υ in (4.3.92) as the* Cauchy index *of ϕ and we denote it by* $\operatorname{ind}\phi$. *If $\phi \in \mathcal{G}SAP$ is such that $\kappa(\phi_l) = \kappa(\phi_r) = 0$ and $0 \notin [\mathbf{d}(\phi_l), \mathbf{d}(\phi_r)]$ or if $\phi \in \mathcal{G}SAP$ is such that $\kappa(\phi_l) + \kappa(\phi_r) = 0$, $\Re e\big(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)\big) \neq 0$ and $0 \notin [\mathbf{d}(\phi_l), \mathbf{d}(\phi_r)]$, we define* winding number *of ϕ as*

$$\begin{aligned} \operatorname{wind}\phi &:= \operatorname{ind}\phi - \frac{1}{2} + \left\{\frac{1}{2} + \frac{1}{2\pi}\arg\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right\} \\ &= \operatorname{ind}\phi + \frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right\} \\ &= \operatorname{ind}\phi + \frac{1}{2\pi}\arg\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}, \end{aligned} \tag{4.3.103}$$

where the last equality is valid if and only if $\arg(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) \in (-\pi, \pi)$.

We remark that the above condition $0 \notin [\mathbf{d}(\phi_l), \mathbf{d}(\phi_r)]$ is equivalent to the condition $\frac{1}{2\pi}\arg\left(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)\right) \notin \mathbb{Z} + \frac{1}{2}$, being therefore clear the existence of a nonzero fractional part $\{\cdot\}$ in the winding number formula.

As we can see above, the previous notions are more general than the already existent definitions of Cauchy index and winding number in the framework of Fredholm Wiener--Hopf operators (i.e., for $\phi \in \mathcal{G}SAP$ such that $\kappa(\phi_l) = \kappa(\phi_r) = 0$ and $0 \notin [\mathbf{d}(\phi_l), \mathbf{d}(\phi_r)]$). Since Definition 4.3.3 embraces the first definitions of Cauchy index and winding number, we may concluded that Definition 4.3.3 is consistent with the first definitions of Cauchy index and winding number for $\phi \in \mathcal{G}SAP$ with $\kappa(\phi_l) = \kappa(\phi_r) = 0$ and $0 \notin [\mathbf{d}(\phi_l), \mathbf{d}(\phi_r)]$. Concerning strictly the definition of Cauchy index, we may also see that Definition 4.3.3 is consistent with the well-known definition of Cauchy index for functions in $\mathcal{G}C(\overline{\mathbb{R}})$. In this case this occurs because Υ in (4.3.92) is equal to (4.3.90).

After having the Cauchy index and winding number notions in the framework of Definition 4.3.3, we are now in conditions to present a Fredholm index formula for Fredholm Wiener-Hopf-Hankel operators with SAP Fourier symbols.

Theorem 4.3.4. *Let* $\phi \in \mathcal{G}SAP$.

(a) *If* $\kappa(\phi_l) + \kappa(\phi_r) = 0$ *and* $\Re e(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) \neq 0$, *then*

$$\mathrm{Ind}\,(W \pm H)_\phi = -\,\mathrm{wind}\,\rho_\phi; \tag{4.3.104}$$

(b) *if* $\kappa(\phi_l) = \kappa(\phi_r) = 0$ *and* $\frac{1}{2\pi}\arg\left(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)\right) \notin \mathbb{Z} + \frac{1}{4}$, *then*

$$\mathrm{Ind}\,(W + H)_\phi = -\,\mathrm{wind}\,\rho_\phi; \tag{4.3.105}$$

(c) *and, if* $\kappa(\phi_l) = \kappa(\phi_r) = 0$ *and* $\frac{1}{2\pi}\arg\left(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)\right) \notin \mathbb{Z} - \frac{1}{4}$, *then*

$$\mathrm{Ind}\,(W - H)_\phi = -\,\mathrm{wind}\,\rho_\phi, \tag{4.3.106}$$

where

$$\rho_\phi(x) := \phi(x)e^{-\left(1-u(x)\right)\left(i\kappa(\phi_l)x+\omega_l(x)\right)-u(x)\left(i\kappa(\phi_r)x+\omega_r(x)\right)}\mathbf{d}(\phi_l)^{u(x)-1}\,\mathbf{d}(\phi_r)^{-u(x)}. \tag{4.3.107}$$

Moreover, if $0 \notin [\mathbf{d}(\phi_l), \mathbf{d}(\phi_r)]$,

(i) $\kappa(\phi_l) + \kappa(\phi_r) = 0$ *and* $\Re e(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) \neq 0$, *then*

$$\mathrm{Ind}\,(W \pm H)_\phi = -\,\mathrm{wind}\,\phi; \tag{4.3.108}$$

(ii) $\kappa(\phi_l) = \kappa(\phi_r) = 0$ *and* $\frac{1}{2\pi}\arg\left(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)\right) \notin \mathbb{Z} + \frac{1}{4}$, *then*

$$\mathrm{Ind}\,(W + H)_\phi = -\,\mathrm{wind}\,\phi; \tag{4.3.109}$$

(iii) $\kappa(\phi_l) = \kappa(\phi_r) = 0$ *and* $\frac{1}{2\pi}\arg\left(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)\right) \notin \mathbb{Z} - \frac{1}{4}$, *then*

$$\mathrm{Ind}\,(W - H)_\phi = -\,\mathrm{wind}\,\phi. \tag{4.3.110}$$

Proof. From the proof of Theorem 4.1.2, we already know that we may rewrite $\phi \in \mathcal{G}SAP$ as

$$\phi = (1-u)\mathbf{d}(\phi_l)e_{\kappa(\phi_l)}e^{\omega_l} + u\mathbf{d}(\phi_r)e_{\kappa(\phi_r)}e^{\omega_r} + \phi_0, \tag{4.3.111}$$

where $u \in C(\overline{\mathbb{R}})$ for which $u(-\infty) = 0$ and $u(+\infty) = 1$, $\phi_l, \phi_r \in \mathcal{G}AP$ are the almost periodic representatives of ϕ at $-\infty$ and $+\infty$ (respectively), $\phi_0 \in C_0(\dot{\mathbb{R}})$, and $\omega_l, \omega_r \in AP$ such that $M(\omega_l) = M(\omega_r) = 0$. Additionally, without loss of generality, let us assume that $u \in C(\overline{\mathbb{R}})$ is a real-valued function. Consider now the auxiliary function

$$\rho_\phi(x) = \phi(x) e^{-\left(1-u(x)\right)\left(i\kappa(\phi_l)x+\omega_l(x)\right)-u(x)\left(i\kappa(\phi_r)x+\omega_r(x)\right)} \mathbf{d}(\phi_l)^{u(x)-1}\, \mathbf{d}(\phi_r)^{-u(x)}. \tag{4.3.112}$$

Since $\rho_\phi(-\infty) = \rho_\phi(+\infty) = 1$, we have that $\rho_\phi \in C(\dot{\mathbb{R}})$. Moreover, since ϕ is an invertible function, it follows from (4.3.112) that $\rho_\phi \in \mathcal{G}C(\dot{\mathbb{R}})$. Because of this, H_{ρ_ϕ} is a compact operator and W_{ρ_ϕ} is a Fredholm operator with

$$\operatorname{Ind} W_{\rho_\phi} = -\operatorname{wind} \rho_\phi \tag{4.3.113}$$

(see Theorem 4.3.1). Due to the compactness of operator H_{ρ_ϕ} and in virtue of W_{ρ_ϕ} being a Fredholm operator, it results that $(W\pm H)_{\rho_\phi}$ are Fredholm operators and

$$\operatorname{Ind}(W\pm H)_{\rho_\phi} = \operatorname{Ind} W_{\rho_\phi} = -\operatorname{wind} \rho_\phi. \tag{4.3.114}$$

For $\lambda \in [0,1]$, define

$$\phi_{\lambda}(x) := \rho_\phi(x)\, e^{\lambda\left(\left(1-u(x)\right)\left(i\kappa(\phi_l)x+\omega_l(x)\right)+u(x)\left(i\kappa(\phi_r)x+\omega_r(x)\right)\right)} \mathbf{d}(\phi_l)^{\left(1-(u(x))^{\lambda}\right)}\, \mathbf{d}(\phi_r)^{(u(x))^{\lambda}}. \tag{4.3.115}$$

Notice that $\phi_{\lambda} \in \mathcal{G}SAP$, for all $\lambda \in [0,1]$. In particular, $\phi_{0} = \rho_\phi\, \mathbf{d}(\phi_r)$ and $\phi_{1} = \phi$. Since

$$(W\pm H)_{\rho_\phi \mathbf{d}(\phi_r)} = \mathbf{d}(\phi_r)(W\pm H)_{\rho_\phi}, \tag{4.3.116}$$

we have that $(W\pm H)_{\rho_\phi \mathbf{d}(\phi_r)}$ are also Fredholm operators with a Fredholm index given by

$$\operatorname{Ind}(W\pm H)_{\rho_\phi \mathbf{d}(\phi_r)} = \operatorname{Ind}(W\pm H)_{\rho_\phi} = -\operatorname{wind} \rho_\phi. \tag{4.3.117}$$

At this point, we already know that $(W\pm H)_{\phi_0}$ and $(W\pm H)_{\phi_1}$ are Fredholm operators and we will now study this property for $(W\pm H)_{\phi_\lambda}$, with $\lambda \in (0,1)$. The almost periodic representatives of ϕ_{λ} (for $\lambda \in (0,1)$) are

$$\left(\phi_{\lambda}\right)_l = e_{\lambda\kappa(\phi_l)} e^{\lambda\omega_l} \mathbf{d}(\phi_l), \quad \left(\phi_{\lambda}\right)_r = e_{\lambda\kappa(\phi_r)} e^{\lambda\omega_r} \mathbf{d}(\phi_r). \tag{4.3.118}$$

Therefore, we get $\kappa\big((\phi_\lambda)_l\big) = \lambda\kappa(\phi_l)$ and $\kappa\big((\phi_\lambda)_r\big) = \lambda\kappa(\phi_r)$. Additionally, since $M(\omega_l) = M(\omega_r) = 0$, we also have $\mathbf{d}\big((\phi_\lambda)_l\big) = \mathbf{d}(\phi_l)$ and $\mathbf{d}\big((\phi_\lambda)_r\big) = \mathbf{d}(\phi_r)$. By hypothesis, it follows that: $\kappa\big((\phi_\lambda)_l\big) + \kappa\big((\phi_\lambda)_r\big) = 0$ and $\Re e\big(\mathbf{d}\big((\phi_\lambda)_l\big)/\mathbf{d}\big((\phi_\lambda)_r\big)\big) \neq 0$ for the conditions of case (a); $\kappa\big((\phi_\lambda)_l\big) = \kappa\big((\phi_\lambda)_r\big) = 0$ and $\frac{1}{2\pi}\arg\big(\big(\mathbf{d}(\phi_\lambda)_l\big)/\mathbf{d}\big((\phi_\lambda)_r\big)\big) \notin \mathbb{Z} + \frac{1}{4}$ for the conditions of case (b); and $\kappa\big((\phi_\lambda)_l\big) = \kappa\big((\phi_\lambda)_r\big) = 0$ and $\frac{1}{2\pi}\arg\big(\big(\mathbf{d}(\phi_\lambda)_l\big)/\mathbf{d}\big((\phi_\lambda)_r\big)\big) \notin \mathbb{Z} - \frac{1}{4}$ for the conditions of case (c). Thus, from Theorem 4.1.9, we conclude for all $\lambda \in (0,1)$ that $(W \pm H)_{\phi_\lambda}$, $(W+H)_{\phi_\lambda}$ and $(W-H)_{\phi_\lambda}$ are Fredholm operators in case (a), (b) and (c), respectively. Consequently, since the maps

$$[0,1] \to \mathcal{B}(L^2_+(\mathbb{R}), L^2(\mathbb{R}_+)), \;\; \lambda \mapsto (W+H)_{\phi_\lambda}, \tag{4.3.119}$$

$$[0,1] \to \mathcal{B}(L^2_+(\mathbb{R}), L^2(\mathbb{R}_+)), \;\; \lambda \mapsto (W-H)_{\phi_\lambda} \tag{4.3.120}$$

are continuous, applying the homotopy argument (see, e.g., [54, Theorem 3.11]), we obtain

$$\operatorname{Ind}(W \pm H)_{\phi_1} = \operatorname{Ind}(W \pm H)_{\phi_0}\,, \tag{4.3.121}$$

i.e.,

$$\operatorname{Ind}(W \pm H)_{\phi} = \operatorname{Ind}(W \pm H)_{\rho_\phi\, \mathbf{d}(\phi_r)}\,. \tag{4.3.122}$$

Thus, combining (4.3.117) with (4.3.122), it follows that

$$\operatorname{Ind}(W \pm H)_{\phi} = -\operatorname{wind}\rho_\phi \tag{4.3.123}$$

$$\operatorname{Ind}(W + H)_{\phi} = -\operatorname{wind}\rho_\phi \tag{4.3.124}$$

$$\operatorname{Ind}(W - H)_{\phi} = -\operatorname{wind}\rho_\phi, \tag{4.3.125}$$

if in the conditions of case (a), (b) and (c), respectively. Finally, it only remains to prove that $\operatorname{wind}\rho_\phi = \operatorname{wind}\phi$, when $0 \notin [\mathbf{d}(\phi_l), \mathbf{d}(\phi_r)]$, i.e., when the winding number of ϕ is defined. Since $\rho_\phi \in \mathcal{G}C(\dot{\mathbb{R}})$, we have $\operatorname{wind}\rho_\phi = \operatorname{ind}\rho_\phi$. By the definition of Cauchy index, it results that

$$\begin{aligned}\operatorname{ind}\rho_\phi = \operatorname{ind}\phi - \operatorname{ind}\Big(e^{\big(1-u(x)\big)\big(i\kappa(\phi_l)x+\omega_l(x)\big)+u(x)\big(i\kappa(\phi_r)x+\omega_r(x)\big)}\Big) \\ + \operatorname{ind}\Big(\mathbf{d}(\phi_l)^{u(x)-1}\Big) - \operatorname{ind}\Big(\mathbf{d}(\phi_r)^{u(x)}\Big).\end{aligned} \tag{4.3.126}$$

Computing the last two Cauchy indices on the right-hand side of (4.3.126), we obtain

$$\operatorname{ind}\Big(\mathbf{d}(\phi_l)^{u(x)-1}\Big) = \frac{1}{2\pi}\arg(\mathbf{d}(\phi_l)), \tag{4.3.127}$$

$$\operatorname{ind}\Big(\mathbf{d}(\phi_r)^{u(x)}\Big) = \frac{1}{2\pi}\arg(\mathbf{d}(\phi_r)). \tag{4.3.128}$$

Concerning the Cauchy index of $e^{\big(1-u(x)\big)\big(i\kappa(\phi_l)x+\omega_l(x)\big)+u(x)\big(i\kappa(\phi_r)x+\omega_r(x)\big)}$, we need first to obtain a formula for the continuous argument

$$\Xi(x) = \arg\Big(e^{\big(1-u(x)\big)\big(i\kappa(\phi_l)x+\omega_l(x)\big)+u(x)\big(i\kappa(\phi_r)x+\omega_r(x)\big)}\Big). \tag{4.3.129}$$

Taking into account that u is a real-valued function, we have

$$\begin{aligned}\Xi(x) &= \Im m\Big(\big(1-u(x)\big)\big(i\kappa(\phi_l)x+\omega_l(x)\big)+u(x)\big(i\kappa(\phi_r)x+\omega_r(x)\big)\Big)+2k\pi\\ &= \big(1-u(x)\big)\big(\kappa(\phi_l)x+\Im m\,\omega_l(x)\big)+u(x)\big(\kappa(\phi_r)x+\Im m\,\omega_r(x)\big)+2k\pi, \quad k\in\mathbb{Z}.\end{aligned} \tag{4.3.130}$$

Thus, by the definition of the Cauchy index (cf. Definition 4.3.3) and using the hypothesis $\kappa(\phi_l)+\kappa(\phi_r)=0$, it follows

$$\begin{aligned}&\operatorname{ind}\Big(e^{\big(1-u(x)\big)\big(i\kappa(\phi_l)x+\omega_l(x)\big)+u(x)\big(i\kappa(\phi_r)x+\omega_r(x)\big)}\Big)\\ &= \frac{1}{2\pi}\lim_{T\to+\infty}\frac{1}{T}\int_T^{2T}\Big[\big(1-u(x)\big)\big(\kappa(\phi_l)x+\Im m\,\omega_l(x)\big)\\ &\qquad + u(x)\big(\kappa(\phi_r)x+\Im m\,\omega_r(x)\big)-\big(1-u(-x)\big)\big(-\kappa(\phi_l)x+\Im m\,\omega_l(-x)\big)\\ &\qquad - u(-x)\big(-\kappa(\phi_r)x+\Im m\,\omega_r(-x)\big)\Big]dx\\ &= \frac{1}{2\pi}\lim_{T\to+\infty}\frac{1}{T}\int_T^{2T}\Big[\kappa(\phi_r)x+\Im m\,\omega_r(x)+\kappa(\phi_l)x-\Im m\,\omega_l(-x)\Big]dx\\ &= \frac{1}{2\pi}\lim_{T\to+\infty}\frac{1}{T}\int_T^{2T}\Big[\Im m\,\omega_r(x)-\Im m\,\omega_l(-x)\Big]dx\\ &= \frac{1}{2\pi}\Big[M(\Im m\,\omega_r)-M(\Im m\,\omega_l)\Big].\end{aligned} \tag{4.3.131}$$

Because $\omega_l,\omega_r\in AP$ are such that $M(\omega_l)=M(\omega_r)=0$, then $M(\Im m\,\omega_r)=M(\Im m\,\omega_l)=0$. Consequently,

$$\operatorname{ind}\Big(e^{\big(1-u(x)\big)\big(i\kappa(\phi_l)x+\omega_l(x)\big)+u(x)\big(i\kappa(\phi_r)x+\omega_r(x)\big)}\Big)=0. \tag{4.3.132}$$

Combining (4.3.126), (4.3.127), (4.3.128) and (4.3.132), it results that

$$\begin{aligned} \operatorname{ind}\rho_\phi &= \operatorname{ind}\phi \; + \frac{1}{2\pi}\Big(\arg(\mathbf{d}(\phi_l)) - \arg(\mathbf{d}(\phi_r))\Big) \\ &= \operatorname{ind}\phi \; + \frac{1}{2\pi}\arg\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)} \\ &= \operatorname{wind}\phi, \end{aligned} \tag{4.3.133}$$

since $\arg\left(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)\right) \in (-\pi,\pi)$. Recalling that $\operatorname{wind}\rho_\phi = \operatorname{ind}\rho_\phi$ (because $\rho_\phi \in \mathcal{G}C(\dot{\mathbb{R}})$), we finally obtain

$$\operatorname{wind}\rho_\phi = \operatorname{wind}\phi\,. \tag{4.3.134}$$

□

Remark 4.3.5. The key of the Fredholm index formula presented above is the continuous function ρ_ϕ. This function plays an important role since $\rho_\phi \in \mathcal{G}C(\dot{\mathbb{R}})$ is such that $\operatorname{Ind}(W\pm H)_{\rho_\phi} = \operatorname{Ind}(W\pm H)_\phi$ and $\operatorname{wind}\rho_\phi = \operatorname{wind}\phi$. In this way, we obtain the Fredholm index of the Wiener-Hopf-Hankel operator expressed in terms of the winding number of a continuous function for which we can relate with the winding number of the SAP Fourier symbol of the operator (whenever the winding number of the Fourier symbol of the operator is defined). Due to the requirements mentioned above, the obtainment of an expression for ρ_ϕ turned out to be a difficult task. Although we have inspired in the proofs given by D. Sarason in [67, Theorem 1] and also by A. Böttcher, Yu. I. Karlovich and I. M. Spitkovsky in [12, Theorem 3.14], the inclusion of the factor $\mathbf{d}(\phi_l)^{u(x)-1}\,\mathbf{d}(\phi_r)^{-u(x)}$ in the expression of ρ_ϕ was not so intuitive, and therefore several attempts were made till reach to this conclusion.

As a consequence of the Fredholm index formula existent for Fredholm Wiener-Hopf operators with semi-almost periodic Fourier symbols and of the Δ–relation after extension between Wiener-Hopf and Wiener-Hopf plus Hankel operators, we provide in the next theorem a condition for the invertibility of Wiener-Hopf-Hankel operators based on the winding number of the corresponding Fourier symbols.

Theorem 4.3.6. *If $\phi \in \mathcal{G}SAP$ is such that $\kappa(\phi_l)+\kappa(\phi_r)=0$, $\Re e(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) > 0$ and $\operatorname{wind}\phi = 0$, then the Wiener-Hopf-Hankel operators $(W\pm H)_\phi$ are invertible.*

Proof. Consider the operator $W_{\widetilde{\phi\phi^{-1}}}$. As we have seen before (in the proof of Theorem 4.1.2), since $\phi \in \mathcal{G}SAP$ is such that $\kappa(\phi_l) + \kappa(\phi_r) = 0$ and $\Re e(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) \neq 0$, it follows that $W_{\widetilde{\phi\phi^{-1}}}$ is a Fredholm operator. Due to the *Coburn-Simonenko Theorem* [12, Theorem 2.5 and Corollary 2.6], it holds that $W_{\widetilde{\phi\phi^{-1}}}$ is an invertible operator if and only if $\operatorname{Ind} W_{\widetilde{\phi\phi^{-1}}} = 0$. Using the hypothesis $\operatorname{wind}\phi = 0$, we will show that $\operatorname{Ind} W_{\widetilde{\phi\phi^{-1}}} = 0$, and therefore, we have proved that $W_{\widetilde{\phi\phi^{-1}}}$ is an invertible operator. So, applying the Fredholm index formula for Wiener-Hopf operators with SAP Fourier symbols (see Theorem 4.3.2) and the definition of winding number, we have

$$\operatorname{Ind} W_{\widetilde{\phi\phi^{-1}}} = -\operatorname{wind}(\widetilde{\phi\phi^{-1}}) \tag{4.3.135}$$

$$= -\operatorname{ind}(\widetilde{\phi\phi^{-1}}) - \frac{1}{2\pi} \arg \frac{\mathbf{d}\Big((\widetilde{\phi\phi^{-1}})_l\Big)}{\mathbf{d}\Big((\widetilde{\phi\phi^{-1}})_r\Big)}, \tag{4.3.136}$$

where $\arg \mathbf{d}((\widetilde{\phi\phi^{-1}})_l)/\mathbf{d}((\widetilde{\phi\phi^{-1}})_r) \in (-\pi, \pi)$. Additionally, recalling that $(\arg \widetilde{\phi})(x) = (\arg \phi)(-x)$ and $(\arg \phi^{-1})(x) = -(\arg \phi)(x)$, it holds that

$$\begin{aligned}\Big(\arg(\widetilde{\phi\phi^{-1}})\Big)(x) &= (\arg \phi)(x) + \Big(\arg \widetilde{\phi^{-1}}\Big)(x) + 2k\pi \\ &= (\arg\phi)(x) - (\arg\phi)(-x) + 2k\pi, \quad k \in \mathbb{Z}.\end{aligned} \tag{4.3.137}$$

Thus, from the definition of Cauchy index (cf. Definition 4.3.3 and (4.3.92)), and considering, for an unbounded set $A \subset \mathbb{R}_+$, a family of intervals $I_\alpha \subset \mathbb{R}$, $\{I_\alpha\}_{\alpha\in A} = \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$, such that $|I_\alpha| = y_\alpha - x_\alpha \to \infty$, as $\alpha \to \infty$, it follows that

$$\begin{aligned}\operatorname{ind}(\widetilde{\phi\phi^{-1}}) &= \frac{1}{2\pi} \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} \Big(\Big(\arg(\widetilde{\phi\phi^{-1}})\Big)(x) - \Big(\arg(\widetilde{\phi\phi^{-1}})\Big)(-x)\Big)\, dx \\ &= \frac{1}{2\pi} \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} \Big((\arg\phi)(x) - (\arg\phi)(-x) - \Big((\arg\phi)(-x) - (\arg\phi)(x)\Big)\Big)\, dx \\ &= \frac{1}{2\pi} \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} 2\Big((\arg\phi)(x) - (\arg\phi)(-x)\Big) dx \\ &= 2 \operatorname{ind}\phi.\end{aligned} \tag{4.3.138}$$

Relatively to $\arg\Big(\mathbf{d}((\widetilde{\phi\phi^{-1}})_l)/\mathbf{d}((\widetilde{\phi\phi^{-1}})_r)\Big)$, using the following identities from the proof of

Theorem 4.1.2,

$$\mathbf{d}\big((\phi\widetilde{\phi^{-1}})_l\big) = \mathbf{d}(\phi_l)/\mathbf{d}(\phi_r), \tag{4.3.139}$$

$$\mathbf{d}\big((\phi\widetilde{\phi^{-1}})_r\big) = \mathbf{d}(\phi_r)/\mathbf{d}(\phi_l)\,, \tag{4.3.140}$$

it holds

$$\arg \frac{\mathbf{d}\Big(\big(\phi\widetilde{\phi^{-1}}\big)_l\Big)}{\mathbf{d}\Big(\big(\phi\widetilde{\phi^{-1}}\big)_r\Big)} = \arg\left(\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right)^2\right). \tag{4.3.141}$$

Moreover, considering $\arg\left(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)\right) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, it follows that

$$\arg\left(\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right)^2\right) = 2\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right), \tag{4.3.142}$$

i.e.,

$$\arg \frac{\mathbf{d}\Big(\big(\phi\widetilde{\phi^{-1}}\big)_l\Big)}{\mathbf{d}\Big(\big(\phi\widetilde{\phi^{-1}}\big)_r\Big)} = 2\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right). \tag{4.3.143}$$

Noticing that $\Re e(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) > 0$ is equivalent to $-\frac{1}{2}+2\kappa\pi < \arg\left(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)\right) < \frac{1}{2}+2\kappa\pi$ $(\kappa \in \mathbb{Z})$, which implies that $\frac{1}{2\pi}\arg\left(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)\right) \notin \mathbb{Z}+\frac{1}{2}$, i.e., $0 \notin [\mathbf{d}(\phi_l), \mathbf{d}(\phi_r)]$, we have the winding number of ϕ well defined. Combining (4.3.138) and (4.3.143) with the definition of winding number of $\phi\widetilde{\phi^{-1}}$ (cf. (4.3.136)), it results that

$$\operatorname{wind}(\phi\widetilde{\phi^{-1}}) = 2\operatorname{wind}\phi. \tag{4.3.144}$$

Finally, from the last identity and (4.3.135), it yields that

$$\operatorname{Ind} W_{\phi\widetilde{\phi^{-1}}} = -2\operatorname{wind}\phi. \tag{4.3.145}$$

Since, by hypothesis, $\operatorname{wind}\phi = 0$, it follows that $\operatorname{Ind} W_{\phi\widetilde{\phi^{-1}}} = 0$. Therefore, due to the equivalence between the invertibility and the zero Fredholm index properties of the Fredholm operator $W_{\phi\widetilde{\phi^{-1}}}$, it turns out that $W_{\phi\widetilde{\phi^{-1}}}$ is an invertible operator. Taking into consideration Corollary 1.3.8 and Corollary 1.3.10, we conclude that $(W \pm H)_\phi$ are invertible operators. □

Theorem 4.3.7. *If $\phi \in \mathcal{G}SAP$ is such that $\kappa(\phi_l) + \kappa(\phi_r) = 0$, $\Re e(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) < 0$ and $\Im m(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) \neq 0$, then the Wiener-Hopf-Hankel operators $(W{\pm}H)_\phi$ are non invertible Fredholm operators.*

Proof. From Theorem 4.1.9 and from the proof of Theorem 4.1.2, it follows that $(W{\pm}H)_\phi$ and $W_{\phi\widetilde{\phi^{-1}}}$ are Fredholm operators. Moreover, from the hypothesis $\Re e(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) < 0$ and $\Im m(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) \neq 0$, and considering the principal value of $\arg(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r))$ (i.e., $\arg(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) \in (-\pi, \pi)$), we have the following cases: (a) $-\pi < \arg(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) < -\frac{\pi}{2}$; and, (b) $\frac{\pi}{2} < \arg(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) < \pi$. From the definition of winding number of a SAP function, it holds that

$$\operatorname{wind}(\phi\widetilde{\phi^{-1}}) = \operatorname{ind}(\phi\widetilde{\phi^{-1}}) - \frac{1}{2\pi} \arg \frac{\mathbf{d}\Big(\big(\phi\widetilde{\phi^{-1}}\big)_l\Big)}{\mathbf{d}\Big(\big(\phi\widetilde{\phi^{-1}}\big)_r\Big)}, \tag{4.3.146}$$

where $\arg \mathbf{d}((\phi\widetilde{\phi^{-1}})_l)/\mathbf{d}((\phi\widetilde{\phi^{-1}})_r) \in (-\pi, \pi)$. In order to express the winding number of $\phi\widetilde{\phi^{-1}}$ in terms of the winding number of ϕ, and since by (4.3.138) we already know that $\operatorname{ind}(\phi\widetilde{\phi^{-1}}) = 2 \operatorname{ind}\phi$, let us now see how we can relate $\arg \mathbf{d}((\phi\widetilde{\phi^{-1}})_l)/\mathbf{d}((\phi\widetilde{\phi^{-1}})_r)$ with $\arg(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r))$. Since in (4.3.141) we showed that $\arg \mathbf{d}((\phi\widetilde{\phi^{-1}})_l)/\mathbf{d}((\phi\widetilde{\phi^{-1}})_r) = \arg\big((\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r))^2\big)$ and taking into account that $\arg \mathbf{d}((\phi\widetilde{\phi^{-1}})_l)/\mathbf{d}((\phi\widetilde{\phi^{-1}})_r) \in (-\pi, \pi)$, it holds that

(i) if $-\pi < \arg(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) < -\frac{\pi}{2}$, then

$$\arg\left(\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right)^2\right) = 2\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) + 2\pi; \tag{4.3.147}$$

(ii) if $\frac{\pi}{2} < \arg(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) < \pi$, then

$$\arg\left(\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right)^2\right) = 2\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) - 2\pi. \tag{4.3.148}$$

Combining (4.3.147) and (4.3.148) with (4.3.138) and (4.3.146), it results that

(I) if $-\pi < \arg(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) < -\frac{\pi}{2}$, then

$$\operatorname{wind}(\phi\widetilde{\phi^{-1}}) = 2\operatorname{wind}\phi + 1; \tag{4.3.149}$$

(II) if $\frac{\pi}{2} < \arg\left(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)\right) < \pi$, then

$$\mathrm{wind}(\phi\widetilde{\phi^{-1}}) = 2\,\mathrm{wind}\,\phi - 1. \tag{4.3.150}$$

In both cases, we can conclude that $\mathrm{wind}(\phi\widetilde{\phi^{-1}}) \neq 0$ since $\mathrm{wind}\,\phi$ is an integer number. Therefore, due to the Coburn-Simonenko Theorem (see [12, Theorem 2.5 and Corollary 2.6]), it follows that $W_{\phi\widetilde{\phi^{-1}}}$ is not an invertible operator. Finally, using Corollary 1.3.8 and Corollary 1.3.10, it holds that $(W \pm H)_\phi$ are not invertible operators. □

We end up this section presenting two examples that illustrate Theorem 4.3.6 and Theorem 4.3.7.

Example 4.3.8. Consider the function ϕ (see Figure 4.2) given by

$$\phi(x) = (1 - u(x))\, e^{e^{i\pi x}} + u(x)\left(\frac{1}{4} + 2i\right) + \phi_0(x), \tag{4.3.151}$$

with

$$u(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x) \quad \text{and} \quad \phi_0(x) = \frac{1}{1 + \frac{x^2}{100}}. \tag{4.3.152}$$

From Theorem 4.3.6, we conclude that $(W \pm H)_\phi$ are invertible operators.

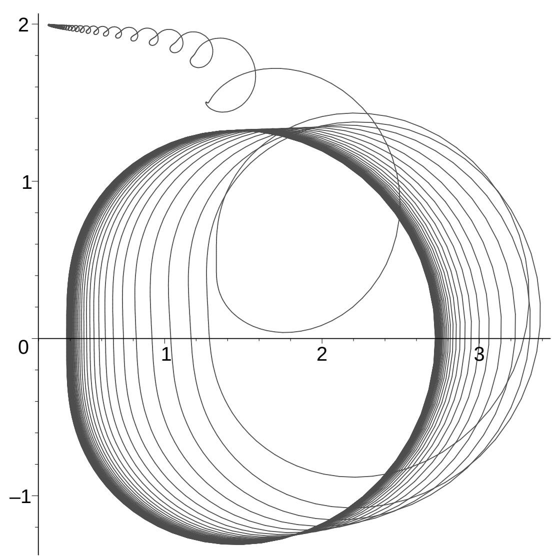


Figure 4.2: The range of $\phi(x)$ defined in (4.3.151) for x between -100 and 100.

Example 4.3.9. Consider now the function ϕ (see Figure 4.3) given by

$$\phi(x) = (1 - u(x))\,(-1 + i) + u(x)\,(-i)\,e^{e^{ix}} + \phi_0(x), \tag{4.3.153}$$

with

$$u(x) = \frac{1}{\pi}\,\mathrm{arccot}(-x) \quad \text{ and } \quad \phi_0(x) = e^{-\frac{|x|}{100}}. \tag{4.3.154}$$

According to Theorem 4.3.7, $(W{\pm}H)_\phi$ are non invertible Fredholm operators. In fact, from Theorem 4.3.4 it holds that

$$\mathrm{Ind}\,(W{\pm}H)_\phi = -\,\mathrm{wind}\,\rho_\phi, \tag{4.3.155}$$

where

$$\rho_\phi(x) = \phi(x)e^{-u(x)e^{ix}}\,(-1+i)^{u(x)-1}\,(-i)^{-u(x)}. \tag{4.3.156}$$

From the graph of ρ_ϕ (cf. Figure 4.4), we obtain $\mathrm{wind}\,\rho_\phi = 1$. Consequently, it follows that $\mathrm{Ind}\,(W{\pm}H)_\phi = -1$, which confirms that $(W{\pm}H)_\phi$ are not invertible operators.

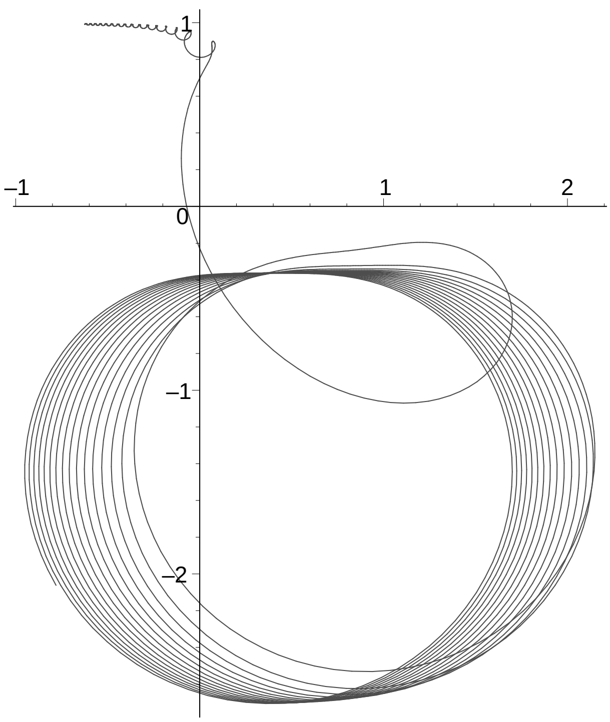


Figure 4.3: The range of $\phi(x)$ defined in (4.3.153) for x between -100 and 100.

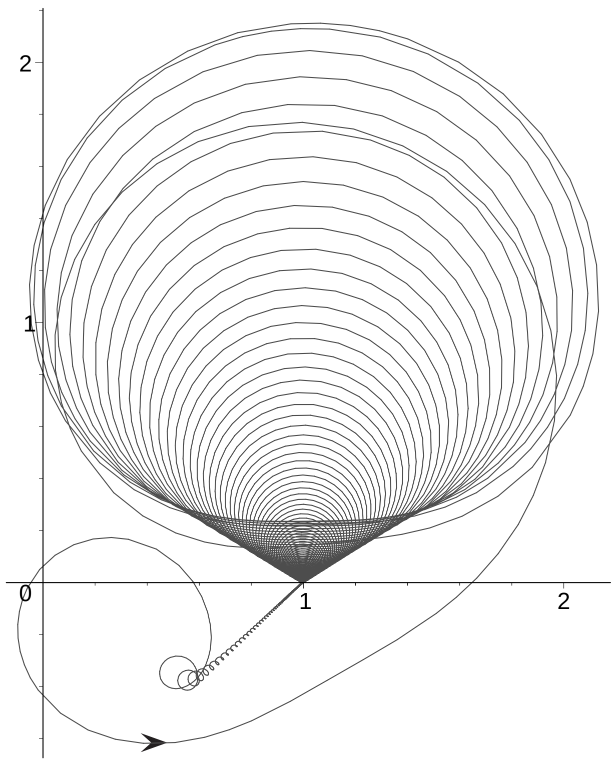


Figure 4.4: The oriented graph of $\rho_\phi(x)$ defined in (4.3.156) for x between -500 and 500.

4.3.2 Fredholm index formula for Wiener-Hopf-Hankel operators with SAP_p Fourier symbols

Recall that, from the Duduchava-Saginashvili's type theorem presented in Theorem 4.2.3, $(W \pm H)_\phi$ are Fredholm operators if $\kappa(\phi_l) + \kappa(\phi_r) = 0$ and $0 \notin \mathcal{A}_p\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}, \frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right)$. In this sense, in the obtainment of the Fredholm index formula, we will only consider Wiener-Hopf--Hankel operators with Fourier symbols belonging to $\mathcal{G}SAP_p$ and satisfying the conditions mentioned above. For that purpose, consider $\phi \in \mathcal{G}SAP_p$. From the proof of Theorem 4.2.3, we have

$$\phi = (1-u)\,\mathbf{d}(\phi_l)e_{\kappa(\phi_l)}e^{\omega_l} + u\,\mathbf{d}(\phi_r)e_{\kappa(\phi_r)}e^{\omega_r} + \phi_0, \tag{4.3.157}$$

where u is a monotonically increasing real-valued function in $C(\overline{\mathbb{R}})$ for which $u(-\infty) = 0$ and $u(+\infty) = 1$, $\phi_l, \phi_r \in \mathcal{G}AP_p$, $\phi_0 \in C_p(\dot{\mathbb{R}})$ such that $\phi_0(\infty) = 0$, and $\omega_l, \omega_r \in AP_p$

satisfying $M(\omega_l) = M(\omega_r) = 0$. We introduce here the function $\rho_\phi \in C_p(\dot{\mathbb{R}})$ given by

$$\rho_\phi(x) := \phi(x) e^{-\left(1-u(x)\right)\left(i\kappa(\phi_l)x+\omega_l(x)\right)-u(x)\left(i\kappa(\phi_r)x+\omega_r(x)\right)} \mathbf{d}(\phi_l)^{u(x)-1}\, \mathbf{d}(\phi_r)^{-u(x)}. \quad (4.3.158)$$

Using this function and recalling that $C_p(\dot{\mathbb{R}}) \subset C(\dot{\mathbb{R}})$, we will reduce the problem of determining the Fredholm index of a Fredholm Wiener-Hopf-Hankel operator with SAP_p Fourier symbol to the computation of the Cauchy index of a continuous function. As we will see in the next result, there is dependency of the Lebesgue index of the spaces, p, but this dependency only appears in the hypothesis. Although the winding number of ρ_ϕ is defined in the context of the p–index notion introduced by I. C. Gohberg and N. Ya. Krupnik [38], the independency of p in the Fredholm index formula occurs due to the fact that the p–index of a continuous function on a closed contour does not depend on p and coincides with its Cauchy index.

Theorem 4.3.10. *Let $\phi \in \mathcal{G}SAP_p$. If $\kappa(\phi_l) + \kappa(\phi_r) = 0$ and $0 \notin \mathcal{A}_p\left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}, \frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right)$, then*

$$\operatorname{Ind}(W \pm H)_\phi = -\operatorname{wind}\rho_\phi. \quad (4.3.159)$$

Proof. Since $\rho_\phi \in \mathcal{G}C_p(\dot{\mathbb{R}})$, it follows that H_{ρ_ϕ} is a compact operator and W_{ρ_ϕ} is a Fredholm operator with

$$\operatorname{Ind} W_{\rho_\phi} = -\operatorname{wind}\rho_\phi. \quad (4.3.160)$$

Consequently, $(W \pm H)_{\rho_\phi}$ are Fredholm operators with

$$\operatorname{Ind}(W \pm H)_{\rho_\phi} = \operatorname{Ind} W_{\rho_\phi}. \quad (4.3.161)$$

Therefore, from (4.3.160), it holds that

$$\operatorname{Ind}(W \pm H)_{\rho_\phi} = -\operatorname{wind}\rho_\phi. \quad (4.3.162)$$

For $\lambda \in [0, 1]$, define now the auxiliary functions

$$\phi_\lambda(x) = \rho_\phi(x)\, e^{\lambda\left(\left(1-u(x)\right)\left(i\kappa(\phi_l)x+\omega_l(x)\right)+u(x)\left(i\kappa(\phi_r)x+\omega_r(x)\right)\right)} \mathbf{d}(\phi_l)^{\left(1-(u(x))^\lambda\right)}\, \mathbf{d}(\phi_r)^{(u(x))^\lambda}. \quad (4.3.163)$$

As we can see, $\phi_\lambda \in \mathcal{G}SAP_p$, for all $\lambda \in [0,1]$, and $\phi_0 = \rho_\phi \, \mathbf{d}(\phi_r)$, $\phi_1 = \phi$. Because

$$(W \pm H)_{\rho_\phi \, \mathbf{d}(\phi_r)} = \mathbf{d}(\phi_r)(W \pm H)_{\rho_\phi}, \tag{4.3.164}$$

it follows that $(W \pm H)_{\rho_\phi \, \mathbf{d}(\phi_r)}$ are also Fredholm operators with a Fredholm index given by

$$\operatorname{Ind}(W \pm H)_{\rho_\phi \, \mathbf{d}(\phi_r)} = \operatorname{Ind}(W \pm H)_{\rho_\phi} = -\operatorname{wind}\rho_\phi. \tag{4.3.165}$$

Up to now we have reached the conclusion that $(W \pm H)_{\phi_0}$ and $(W \pm H)_{\phi_1}$ are Fredholm operators, and such property remains to be studied for $(W \pm H)_{\phi_\lambda}$, with $\lambda \in (0,1)$. Since for $\lambda \in (0,1)$,

$$\left(\phi_\lambda\right)_l = e_{\lambda\kappa(\phi_l)} e^{\lambda\omega_l} \mathbf{d}(\phi_l)\,, \quad \left(\phi_\lambda\right)_r = e_{\lambda\kappa(\phi_r)} e^{\lambda\omega_r} \mathbf{d}(\phi_r)\,, \tag{4.3.166}$$

we get $\kappa\big((\phi_\lambda)_l\big) = \lambda\kappa(\phi_l)$ and $\kappa\big((\phi_\lambda)_r\big) = \lambda\kappa(\phi_r)$. Additionally, using the fact that $M(\omega_l) = M(\omega_r) = 0$, we also have $\mathbf{d}\big((\phi_\lambda)_l\big) = \mathbf{d}(\phi_l)$ and $\mathbf{d}\big((\phi_\lambda)_r\big) = \mathbf{d}(\phi_r)$. From the hypothesis of the theorem, we have $\kappa\big((\phi_\lambda)_l\big) + \kappa\big((\phi_\lambda)_r\big) = 0$ and

$$0 \notin \mathcal{A}_p\left(\frac{\mathbf{d}\big((\phi_\lambda)_r\big)}{\mathbf{d}\big((\phi_\lambda)_l\big)}, \frac{\mathbf{d}\big((\phi_\lambda)_l\big)}{\mathbf{d}\big((\phi_\lambda)_r\big)}\right). \tag{4.3.167}$$

Thus, from the Duduchava-Saginashvili's type theorem (cf. Theorem 4.2.3(c)) we conclude that $(W \pm H)_{\phi_\lambda}$ are Fredholm operators, for all $\lambda \in (0,1)$. Finally, since the maps

$$[0,1] \to \mathcal{L}(L^p_+(\mathbb{R}), L^p(\mathbb{R}_+)),\ \ \lambda \mapsto (W + H)_{\phi_\lambda}, \tag{4.3.168}$$

$$[0,1] \to \mathcal{L}(L^p_+(\mathbb{R}), L^p(\mathbb{R}_+)),\ \ \lambda \mapsto (W - H)_{\phi_\lambda} \tag{4.3.169}$$

are continuous, we can apply the homotopy argument, and then obtain

$$\operatorname{Ind}(W \pm H)_{\phi_1} = \operatorname{Ind}(W \pm H)_{\phi_0}\,, \tag{4.3.170}$$

i.e.,

$$\operatorname{Ind}(W \pm H)_{\phi} = \operatorname{Ind}(W \pm H)_{\rho_\phi \, \mathbf{d}(\phi_r)}\,. \tag{4.3.171}$$

Combining (4.3.165) and (4.3.171), it follows the announced index formula

$$\operatorname{Ind}(W \pm H)_{\phi} = -\operatorname{wind}\rho_\phi. \tag{4.3.172}$$

□

To illustrate the previous theorem – and to exemplify its importance – we present the Fredholm index for three Fredholm Wiener-Hopf-Hankel operators in L^p Lebesgue spaces, having a Fourier symbol in $\mathcal{G}SAP_p$, in the following cases: (i) $p \geq 2$; (ii) $1 < p \leq 2$; (iii) $p \neq 2$.

Example 4.3.11. Consider the function (see Figure 4.5)

$$\phi(x) = (1 - u(x))\,(1 + i)\,e^{2ix} + u(x)\,5\,e^{-2ix} + \phi_0(x), \tag{4.3.173}$$

where

$$u(x) = \begin{cases} \dfrac{1}{2}\,e^{x} & \text{if } x < 0 \\ 1 - \dfrac{1}{2}\,e^{-x} & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad \phi_0(x) = \begin{cases} \dfrac{5\sin(x)}{x} & \text{if } x \neq 0 \\ 5 & \text{if } x = 0 \end{cases}. \tag{4.3.174}$$

Let $2 \leq p < \infty$. From Figure 4.5, we observe that ϕ is an invertible function, and so we have $\phi \in \mathcal{G}SAP_p$. Therefore, from (4.3.173) and proposition (c) in Theorem 4.2.3, it directly follows that $(W \pm H)_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$ are Fredholm operators because $0 \notin \mathcal{A}_p\Big(\frac{5}{1+i}, \frac{1+i}{5}\Big)$, for all $2 \leq p < \infty$. Additionally, from the oriented graph of ρ_ϕ (cf. Figure 4.6) we have $\operatorname{wind} \rho_\phi = -1$ and using Theorem 4.3.10, we obtain $\operatorname{Ind}(W \pm H)_\phi = 1$.

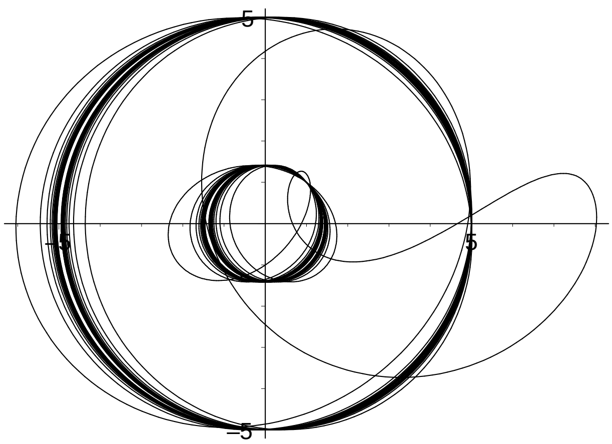


Figure 4.5: The range of $\phi(x)$ defined in (4.3.173) for x between -100 and 100.

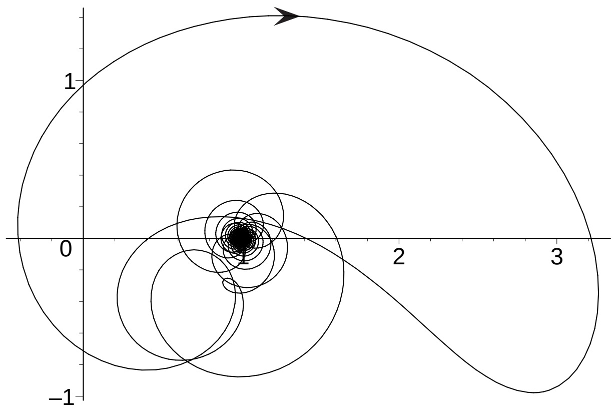


Figure 4.6: The oriented graph of $\rho_\phi(x)$ for x between -1000 and 1000 (cf. Example 4.3.11).

Example 4.3.12. Consider now the function ϕ (see Figure 4.7) given by

$$\phi(x) = (1-u(x))\,(1+i)\,e^{i\pi x} + u(x)\,2i\,e^{-i\pi x} + \phi_0(x), \tag{4.3.175}$$

with

$$u(x) = \frac{1}{2} + \frac{1}{2}\tanh(x) \quad \text{and} \quad \phi_0(x) = \frac{1}{x^2+1}. \tag{4.3.176}$$

Consider $1 < p \leq 2$. From Figure 4.7, it follows that ϕ is an invertible function, and in this way we have $\phi \in \mathcal{G}SAP_p$. Applying proposition (c) in Theorem 4.2.3 to $(W \pm H)_\phi$, and since $0 \notin \mathcal{A}_p\Big(\frac{2i}{1+i}, \frac{1+i}{2i}\Big)$ for all $1 < p \leq 2$, it follows from (4.3.175) that, for $1 < p \leq 2$, $(W \pm H)_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$ are Fredholm operators. Additionally, directly from the graph of ρ_ϕ (cf. Figure 4.8) we observe that $\operatorname{wind}\rho_\phi = 0$. Consequently, Theorem 4.3.10 provides $\operatorname{Ind}(W \pm H)_\phi = 0$.

Example 4.3.13. Consider the function ϕ (see Figure 4.9) given by

$$\phi(x) = (1-u(x))\,i\,e^{i\frac{2}{5}x}\,e^{e^{i\pi x}} + u(x)\;e^{-i\frac{2}{5}x} + \phi_0(x), \tag{4.3.177}$$

where

$$u(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(100x) \quad \text{and} \quad \phi_0(x) = e^{-|x|}. \tag{4.3.178}$$

Let $p \neq 2$. From Figure 4.9, we have that ϕ is an invertible function, and so $\phi \in \mathcal{G}SAP_p$. Since $0 \notin \mathcal{A}_p\Big(\frac{1}{i}, i\Big)$, for all $p \neq 2$, (4.3.177) and proposition (c) in Theorem 4.2.3 assure that $(W \pm H)_\phi : L^p_+(\mathbb{R}) \to L^p(\mathbb{R}_+)$ are Fredholm operators. Moreover, from the graph of ρ_ϕ (cf. Figure 4.10) we have $\operatorname{wind} \rho_\phi = 0$, and according to Theorem 4.3.10 we obtain $\operatorname{Ind}(W \pm H)_\phi = 0$.

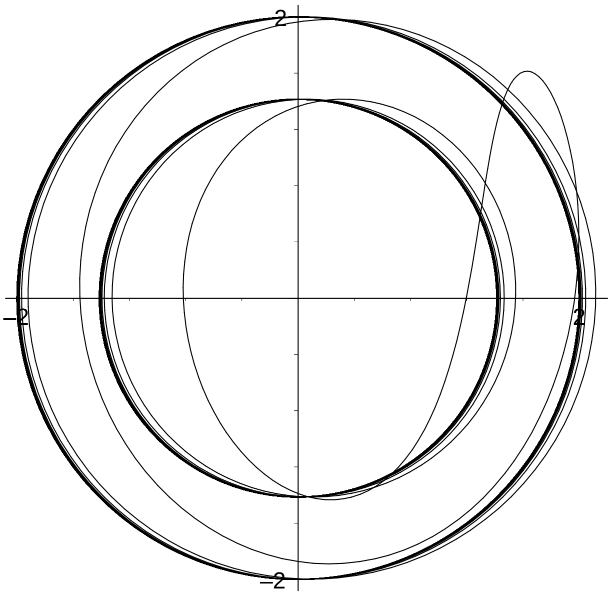


Figure 4.7: The range of $\phi(x)$ defined in (4.3.175) for x between -1000 and 1000.

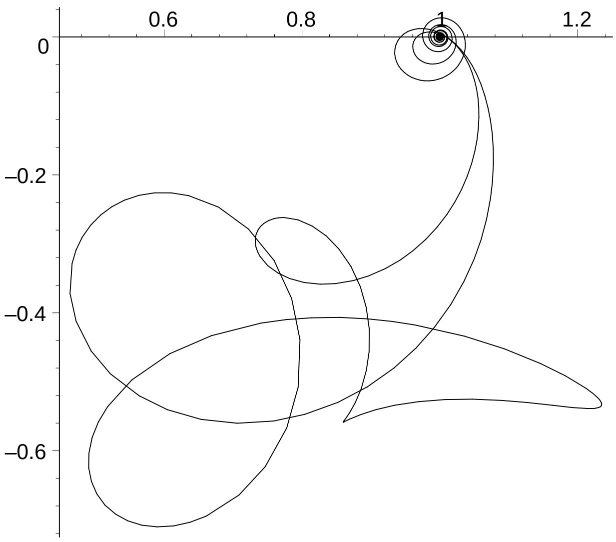


Figure 4.8: The range of $\rho_\phi(x)$ for x between -1000 and 1000 (cf. Example 4.3.12).

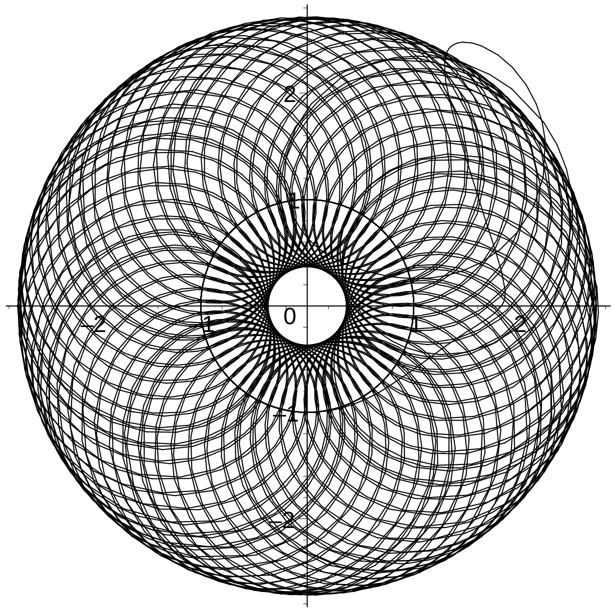


Figure 4.9: The range of $\phi(x)$ defined in (4.3.177) for x between -250 and 250.

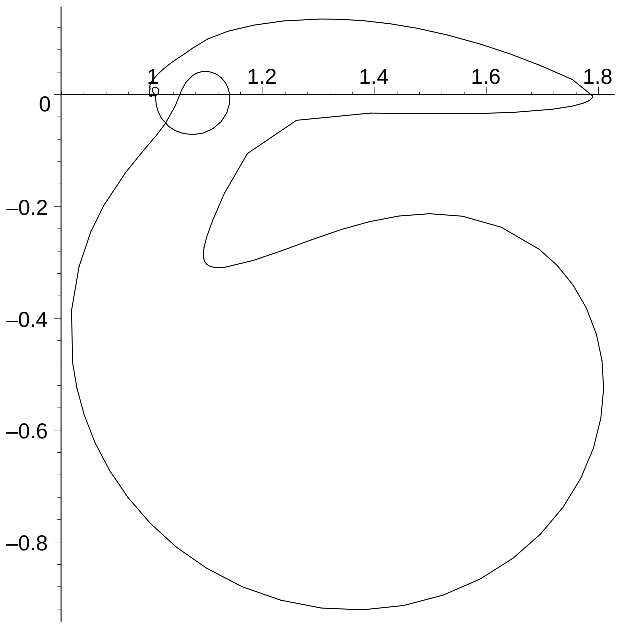


Figure 4.10: The range of $\rho_\phi(x)$ for x between -1000 and 1000 (cf. Example 4.3.13).

Chapter 5

Wiener-Hopf-Hankel Operators with Piecewise Almost Periodic Symbols

In this chapter, we present a generalization of the Sarason's type theorem presented in Section 4.1.2 for Wiener-Hopf-Hankel operators with piecewise almost periodic Fourier symbols and acting between L^2 Lebesgue spaces.

After we reach the conditions for the Fredholmness of the Wiener-Hopf-Hankel operators under study, the natural question of determine a formula for the Fredholm index of Wiener-Hopf-Hankel operators with piecewise almost periodic Fourier symbols arises. Contrarily to the results obtained for Fredholm Wiener-Hopf-Hankel operators with semi--almost periodic Fourier symbols, here we establish a formula for the sum of the Fredholm indices of the Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators. Taking into account the discontinuities of the Fourier symbols, the formula for the sum of the Fredholm indices of the Wiener-Hopf plus and minus Hankel operators will be interpreted and simplified upon different cases of symmetries of the discontinuities of the Fourier symbols. To illustrate this index formula, several examples will be provided. Finally, a condition for the invertibility of Wiener-Hopf-Hankel operators with piecewise almost periodic Fourier symbols is presented in terms of the winding number of the Fourier symbols.

5.1 A semi-Fredholm and invertibility criterion

In recent times, several generalizations of Sarason's Theorem have been made. For instance, in [12] we find the following generalization of Sarason's Theorem for Wiener--Hopf operators with piecewise almost periodic symbols:

Theorem 5.1.1. (cf. [12, Theorem 3.16], and [14, §9.27]) *Let $\phi \in PAP$ such that ϕ is not identically zero.*

(a) *If $\phi \in \mathcal{G}PAP$, $\kappa_l(\phi) = \kappa_r(\phi) = 0$ and*

$$0 \notin \Big[\mathbf{d}_l(\phi), \mathbf{d}_r(\phi)\Big] \cup \bigcup_{x\in\mathbb{R}} \Big[\phi(x-0), \phi(x+0)\Big], \tag{5.1.1}$$

then W_ϕ is a Fredholm operator.

(b) *If $\phi \in \mathcal{G}PAP$, $\kappa_l(\phi) \cdot \kappa_r(\phi) \geq 0$, $\kappa_l(\phi) + \kappa_r(\phi) > 0$ and*

$$0 \notin \bigcup_{x\in\mathbb{R}} \Big[\phi(x-0), \phi(x+0)\Big], \tag{5.1.2}$$

then W_ϕ is properly n-normal and left-invertible.

(c) *If $\phi \in \mathcal{G}PAP$, $\kappa_l(\phi) \cdot \kappa_r(\phi) \geq 0$, $\kappa_l(\phi) + \kappa_r(\phi) < 0$ and*

$$0 \notin \bigcup_{x\in\mathbb{R}} \Big[\phi(x-0), \phi(x+0)\Big], \tag{5.1.3}$$

then W_ϕ is properly d-normal and right-invertible.

(d) *In all the other cases, the operator W_ϕ is not normally solvable.*

Motivated by this last result, we obtain here a "semi-Fredholm theory" for Wiener-Hopf--Hankel operators with piecewise almost periodic Fourier symbols. As we will see below, the addition and the subtraction of the Hankel operator to the Wiener-Hopf operator also introduce in this case several changes in the regularity properties of the resultant operator.

Theorem 5.1.2. *Let $\phi \in \mathcal{G}PAP$.*

(a) *If $\kappa_l(\phi) + \kappa_r(\phi) = 0$ and*

$$0 \notin \left[\frac{\mathbf{d}_l(\phi)}{\mathbf{d}_r(\phi)}, \frac{\mathbf{d}_r(\phi)}{\mathbf{d}_l(\phi)}\right] \cup \bigcup_{x\in\mathbb{R}} \Big[(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)\Big], \tag{5.1.4}$$

then $(W+H)_\phi$ and $(W-H)_\phi$ are Fredholm operators.

(b) *If $\kappa_l(\phi) + \kappa_r(\phi) > 0$ and*

$$0 \notin \bigcup_{x\in\mathbb{R}} \Big[(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)\Big], \tag{5.1.5}$$

then $(W+H)_\phi$ and $(W-H)_\phi$ are left-invertible. Moreover, at least one of these operators is properly n-normal.

(c) *If $\kappa_l(\phi) + \kappa_r(\phi) < 0$ and*

$$0 \notin \bigcup_{x\in\mathbb{R}} \Big[(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)\Big], \tag{5.1.6}$$

then $(W+H)_\phi$ and $(W-H)_\phi$ are right-invertible. In addition, at least one of these operators is properly d-normal.

(d) *In all the remaining cases, at least one of the operators $(W+H)_\phi$ and $(W-H)_\phi$ is not normally solvable.*

Proof. From the definition of PAP (cf. (2.3.34)), we have

$$\phi = (1-u)\phi_l + u\phi_r + \phi_0, \tag{5.1.7}$$

where $\phi_l, \phi_r \in AP$, $\phi_0 \in PC_0$ and $u \in C(\overline{\mathbb{R}})$ satisfying $u(-\infty) = 0$ and $u(+\infty) = 1$. Additionally, since $\phi \in \mathcal{G}PAP$, then $\phi_l, \phi_r \in \mathcal{G}AP$. Applying Bohr's Theorem to ϕ_l, ϕ_r and using then the definition of the geometric mean value, it follows

$$\phi = (1-u)\,\mathbf{d}(\phi_l)e_{\kappa(\phi_l)}e^{\omega_l} + u\,\mathbf{d}(\phi_r)e_{\kappa(\phi_r)}e^{\omega_r} + \phi_0, \tag{5.1.8}$$

with $\omega_l, \omega_r \in AP$ and $M(\omega_l) = M(\omega_r) = 0$.

Once more, we will use the Δ–relation after extension between the Wiener-Hopf plus Hankel operator $(W+H)_\phi$ and the Wiener-Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ to study the regularity properties of the Wiener-Hopf-Hankel operators $(W\pm H)_\phi$ (cf. Corollary 1.3.8 and Corollary 1.3.10). Thus, we will start by studying the regularity properties of the Wiener-Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$, and then we transfer them to the Wiener-Hopf-Hankel operators $(W\pm H)_\phi$. Computing $\phi\widetilde{\phi^{-1}}$, one gets

$$\phi\widetilde{\phi^{-1}} = \frac{(1-u)\,\mathbf{d}(\phi_l)e_{\kappa(\phi_l)}e^{\omega_l} + u\,\mathbf{d}(\phi_r)e_{\kappa(\phi_r)}e^{\omega_r} + \phi_0}{(1-\widetilde{u})\,\mathbf{d}(\phi_l)e_{-\kappa(\phi_l)}e^{\widetilde{\omega_l}} + \widetilde{u}\,\mathbf{d}(\phi_r)e_{-\kappa(\phi_r)}e^{\widetilde{\omega_r}} + \widetilde{\phi_0}}. \tag{5.1.9}$$

By the definition of piecewise almost periodic function, we know that $\phi\widetilde{\phi^{-1}}$ is of the form

$$\phi\widetilde{\phi^{-1}} = (1-v)\big(\phi\widetilde{\phi^{-1}}\big)_l + v\big(\phi\widetilde{\phi^{-1}}\big)_r + \big(\phi\widetilde{\phi^{-1}}\big)_0\,, \tag{5.1.10}$$

where $v \in C(\overline{\mathbb{R}})$ is such that $v(-\infty) = 0$ and $v(+\infty) = 1$, $\big(\phi\widetilde{\phi^{-1}}\big)_0 \in PC_0$ and $\big(\phi\widetilde{\phi^{-1}}\big)_l$, $\big(\phi\widetilde{\phi^{-1}}\big)_r$ are the almost periodic representatives of $\phi\widetilde{\phi^{-1}}$ at $-\infty$ and at $+\infty$, respectively. From (5.1.9), we get

$$\big(\phi\widetilde{\phi^{-1}}\big)_l = \frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\; e_{\kappa(\phi_l)+\kappa(\phi_r)}e^{\omega_l-\widetilde{\omega_r}}\,, \tag{5.1.11}$$

$$\big(\phi\widetilde{\phi^{-1}}\big)_r = \frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}\; e_{\kappa(\phi_l)+\kappa(\phi_r)}\; e^{\omega_r-\widetilde{\omega_l}}\,. \tag{5.1.12}$$

Since $\omega_l, \omega_r \in AP$ are such that $M(\omega_l) = M(\omega_r) = 0$ (recall that in this case $M(\widetilde{\omega_l}) = M(\widetilde{\omega_r}) = 0$), we obtain

$$\kappa\Big(\big(\phi\widetilde{\phi^{-1}}\big)_l\Big) = \kappa\Big(\big(\phi\widetilde{\phi^{-1}}\big)_r\Big) = \kappa(\phi_l) + \kappa(\phi_r) \tag{5.1.13}$$

and

$$\mathbf{d}\Big(\big(\phi\widetilde{\phi^{-1}}\big)_l\Big) = \frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\,, \qquad \mathbf{d}\Big(\big(\phi\widetilde{\phi^{-1}}\big)_r\Big) = \frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)}. \tag{5.1.14}$$

According to the definitions of left and right mean motions, and left and right geometric mean values of piecewise almost periodic functions (see (2.3.37)), it holds

$$\kappa_l\big(\phi\widetilde{\phi^{-1}}\big) = \kappa_r\big(\phi\widetilde{\phi^{-1}}\big) = \kappa_l(\phi) + \kappa_r(\phi) \tag{5.1.15}$$

and

$$\mathbf{d}_l\big(\widetilde{\phi\phi^{-1}}\big) = \frac{\mathbf{d}_l(\phi)}{\mathbf{d}_r(\phi)}, \qquad \mathbf{d}_r\big(\widetilde{\phi\phi^{-1}}\big) = \frac{\mathbf{d}_r(\phi)}{\mathbf{d}_l(\phi)}. \tag{5.1.16}$$

Applying Theorem 5.1.1 to the Wiener-Hopf operator $W_{\widetilde{\phi\phi^{-1}}}$, it follows from (5.1.15) and (5.1.16) that:

(a) if $\kappa_l(\phi) + \kappa_r(\phi) = 0$ and

$$0 \notin \left[\frac{\mathbf{d}_l(\phi)}{\mathbf{d}_r(\phi)}, \frac{\mathbf{d}_r(\phi)}{\mathbf{d}_l(\phi)}\right] \cup \bigcup_{x\in\mathbb{R}} \Big[(\widetilde{\phi\phi^{-1}})(x-0), (\widetilde{\phi\phi^{-1}})(x+0)\Big], \tag{5.1.17}$$

then $W_{\widetilde{\phi\phi^{-1}}}$ is a Fredholm operator;

(b) if $\kappa_l(\phi) + \kappa_r(\phi) > 0$ and

$$0 \notin \bigcup_{x\in\mathbb{R}} \Big[(\widetilde{\phi\phi^{-1}})(x-0), (\widetilde{\phi\phi^{-1}})(x+0)\Big], \tag{5.1.18}$$

then $W_{\widetilde{\phi\phi^{-1}}}$ is properly n-normal and left-invertible;

(c) if $\kappa_l(\phi)+\kappa_r(\phi) < 0$ and $0 \notin \bigcup_{x\in\mathbb{R}} \Big[(\widetilde{\phi\phi^{-1}})(x-0), (\widetilde{\phi\phi^{-1}})(x+0)\Big]$, then $W_{\widetilde{\phi\phi^{-1}}}$ is properly d-normal and right-invertible;

(d) in all the other cases, the operator $W_{\widetilde{\phi\phi^{-1}}}$ is not normally solvable.

Applying now Corollary 1.3.8 and Corollary 1.3.10, we obtain that $(W+H)_\phi$ and $(W-H)_\phi$ are Fredholm operators, left-invertible, and right-invertible, under the conditions of case (a), (b) and (c), respectively. To arrive at the final assertion, we have just to interpret the Δ–relation after extension between the Wiener-Hopf plus Hankel operator $(W+H)_\phi$ and the Wiener-Hopf operator $W_{\widetilde{\phi\phi^{-1}}}$ (cf. Lemma 1.3.7) as an equivalence after extension between $\mathrm{diag}[(W+H)_\phi, \mathcal{T}_\phi]$ and $W_{\widetilde{\phi\phi^{-1}}}$, and use the equivalence after extension between the operators $\mathcal{T}_\phi$ and $(W-H)_\phi$ (cf. Proposition 1.3.9). □

Remark 5.1.3. Concerning the last result, we would like to mention the following:

(a) as we saw before in the proof of Theorem 4.1.2, since $\mathbf{d}_l(\widetilde{\phi\phi^{-1}})$ and $\mathbf{d}_r(\widetilde{\phi\phi^{-1}})$ are inverses of each other (cf. (5.1.16)), condition

$$0 \notin \left[\frac{\mathbf{d}_l(\phi)}{\mathbf{d}_r(\phi)}, \frac{\mathbf{d}_r(\phi)}{\mathbf{d}_l(\phi)}\right] \tag{5.1.19}$$

is satisfied if and only if $\mathbf{d}_l(\phi)/\mathbf{d}_r(\phi)$ is such that $\Re e(\mathbf{d}_l(\phi)/\mathbf{d}_r(\phi)) \neq 0$;

(b) Theorem 5.1.2 may also be called a Sarason's type theorem for Wiener-Hopf-Hankel operators since it describes the Fredholm nature of $(W+H)_\phi$ and $(W-H)_\phi$ based on the values of $\kappa_l(\phi)$, $\kappa_r(\phi)$, $\mathbf{d}_l(\phi)$ and $\mathbf{d}_r(\phi)$ when $\phi \in \mathcal{G}PAP$ and

$$0 \notin \bigcup_{x\in\mathbb{R}} \Big[(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)\Big]; \tag{5.1.20}$$

(c) also in this case we can provide an example that illustrates the differences in the regularity properties of the Wiener-Hopf plus/minus Hankel operators by adding or subtracting the Hankel operator to the Wiener-Hopf operator. If we consider $\phi \in \mathcal{G}PAP$ such that $\kappa_l(\phi)\cdot\kappa_r(\phi) < 0$, $\kappa_l(\phi)+\kappa_r(\phi) \neq 0$,

$$0 \notin \bigcup_{x\in\mathbb{R}} \Big[\phi(x-0), \phi(x+0)\Big] \tag{5.1.21}$$

and

$$0 \notin \bigcup_{x\in\mathbb{R}} \Big[(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)\Big], \tag{5.1.22}$$

then we have that the Wiener-Hopf plus Hankel operator $(W+H)_\phi$ and the Wiener--Hopf minus Hankel operator $(W-H)_\phi$ are normally solvable although the Wiener--Hopf operator W_ϕ is not normally solvable.

5.2 Fredholm index formula

5.2.1 A formula for the sum of the indices of Fredholm Wiener-Hopf plus/minus Hankel operators

In [30], Example 3.25, T. Ehrhardt gave an example of a Toeplitz plus Hankel operator and a Toeplitz minus Hankel operator, with the same Fourier symbol but having different Fredholm indices. The example is the following. For $\beta \in \mathbb{C}$, let

$$t_\beta(e^{i\theta}) := e^{(i\beta(\theta-\pi))}, \quad 0<\theta<2\pi, \tag{5.2.23}$$

and consider Toeplitz and Hankel operators acting on $H^p_+(\mathbb{T})$. If $1/2p < \Re e\beta < 1/2+1/2p$, then $(T+H)_{t_\beta}$ is a Fredholm operator with index -1, while $(T-H)_{t_\beta}$ is an invertible operator and thus has zero Fredholm index.

Considering the isometric isomorphism B_0 from $L^\infty(\mathbb{R})$ onto $L^\infty(\mathbb{T})$ defined in (1.3.118), it is possible to directly construct the corresponding example in the framework of Wiener--Hopf-Hankel operators acting between L^2 Lebesgue spaces. This happens because it is possible to relate, through an equivalence relation, Toeplitz minus Hankel operators with Wiener-Hopf plus Hankel operators and Toeplitz plus Hankel operators with Wiener-Hopf minus Hankel operators (cf. Lemmas 1.3.11 and 1.3.13). In this way, considering

$$\phi_\beta(x) := \left(\frac{x-i}{x+i}\right)^\beta e^{-i\beta\pi}, \quad x \in \mathbb{R}, \tag{5.2.24}$$

with $\beta \in \mathbb{C}$ such that $1/4 < \Re e\beta < 3/4$, the Wiener-Hopf minus Hankel operator $(W-H)_{\phi_\beta} : L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+)$ is a Fredholm operator with $\operatorname{Ind}(W-H)_{\phi_\beta} = -1$, and the Wiener-Hopf plus Hankel operator $(W+H)_{\phi_\beta} : L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+)$ is invertible (and therefore $\operatorname{Ind}(W+H)_{\phi_\beta} = 0$). Figures 5.1 and 5.2 show two Fourier symbols that illustrate this example.

In particular, this example shows that Fredholm Wiener-Hopf plus Hankel and Fredholm Wiener-Hopf minus Hankel operators with piecewise continuous Fourier symbols may have different Fredholm indices. Consequently, the same conclusion holds for Fredholm Wiener-Hopf plus Hankel and Fredholm Wiener-Hopf minus Hankel operators with piecewise almost periodic Fourier symbols, contrarily to what happens in the case of Fredholm Wiener-Hopf-Hankel with semi-almost periodic Fourier symbols where, as we have seen in Theorem 4.3.4 and Theorem 4.3.10, Fredholm Wiener-Hopf plus Hankel and Fredholm Wiener-Hopf minus Hankel operators have the same index.

Due to this, in the present subsection we will achieve a formula for the sum of the indices of Fredholm Wiener-Hopf plus/minus Hankel operators with piecewise almost periodic symbols. Recall that from Theorem 5.1.2, we have that $(W+H)_\phi$ and $(W-H)_\phi$ are

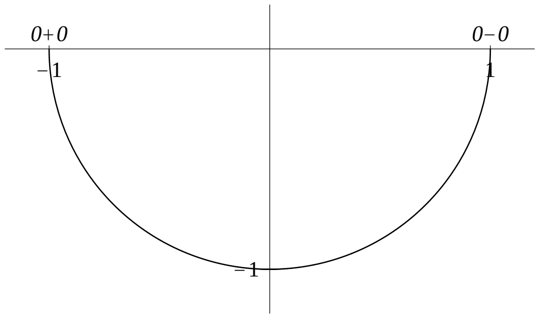


Figure 5.1: The range of $\phi_{\frac{1}{2}}(x)$ defined in (5.2.24) for x between -500 and 500.

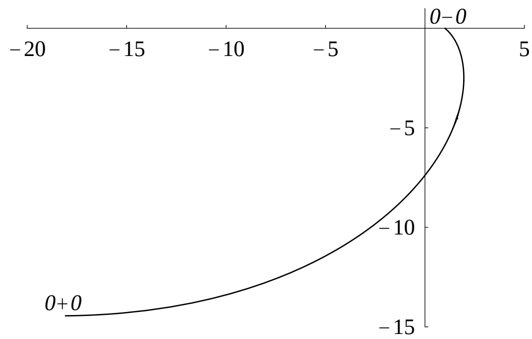


Figure 5.2: The range of $\phi_{\frac{\pi}{8}+\frac{i}{2}}(x)$ defined in (5.2.24) for x between -500 and 500.

Fredholm operators if $\phi \in \mathcal{G}PAP$ is such that $\kappa_l(\phi) + \kappa_r(\phi) = 0$ and

$$0 \notin \left[\frac{\mathbf{d}_l(\phi)}{\mathbf{d}_r(\phi)}, \frac{\mathbf{d}_r(\phi)}{\mathbf{d}_l(\phi)}\right] \cup \bigcup_{x\in\mathbb{R}} \Big[(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)\Big]. \tag{5.2.25}$$

Thus, for all Fourier symbols ϕ satisfying the conditions mentioned above, we will obtain a formula which relates the Fredholm index of the Wiener-Hopf plus Hankel operator $(W+H)_\phi$ with the Fredholm index of the Wiener-Hopf minus Hankel operator $(W-H)_\phi$ based on the winding number of a piecewise almost periodic function (constructed from the initial Fourier symbol of the operators). In addition, taking into account the discontinuities of this piecewise almost periodic function, we will be able to simplify the formula of the winding number of this function accordingly with the following three situations: (1) ϕ has no symmetric discontinuities; (2) ϕ has symmetric discontinuities and ϕ is continuous at 0; and finally, (3) ϕ is discontinuous at 0. We say that $\phi \in PAP$ has *symmetric discontinuities* if ϕ has discontinuities at x_0 and $-x_0$ for some $x_0 \in \mathbb{R}$. Thus, a piecewise almost periodic function ϕ has symmetric discontinuities if and only if ϕ and $\widetilde{\phi}$ have common discontinuities. Moreover, $\phi \in \mathcal{G}PAP$ has symmetric discontinuities if and only if ϕ and $\widetilde{\phi^{-1}}$ have common discontinuities. In particular, a function with a discontinuity at 0 is said to be a function with symmetric discontinuities.

For having all the necessary instruments to establish the Fredholm index formula, we will start with the definitions of winding number and Cauchy index of a piecewise continuous function. Then, we will present the definition of winding number of a piecewise almost periodic function, and finally, we will introduce a generalization of the definition of winding number of a piecewise almost periodic function from the context of Wiener-Hopf operators to the framework of Wiener-Hopf-Hankel operators.

Recall, from Subsection 4.2.2, that for all $\psi \in PC$, the function $\psi^{\#} : \dot{\mathbb{R}} \times [0,1] \to \mathbb{C}$ is defined by

$$\psi^{\#}(x,\mu) = (1-\mu)\,\psi(x-0) + \mu\,\psi(x+0) \tag{5.2.26}$$

such that the range of $\psi^{\#}$ is a continuous closed curve with a natural orientation induced by the orientation of $\mathbb{R}$ from $-\infty$ to $+\infty$. If $\psi^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$, then the *winding number* of ψ is defined as the number of times that the curve $\psi^{\#}\big(\dot{\mathbb{R}}, [0,1]\big)$ surrounds the origin counter-clockwise.

For defining the Cauchy index of a piecewise continuous function, let us first consider $\psi \in PC$ having finitely many jumps. In this case, let $\Lambda_\psi \subset \dot{\mathbb{R}}$ denote the set of points at which ψ is discontinuous, and Θ be the set of all connected components l of $\dot{\mathbb{R}} \backslash \Lambda_\psi$. For each $l \in \Theta$, we define $\mathrm{ind}_l\,\psi$ as $(2\pi)^{-1}$ times the increment of the argument of ψ on l. If $\psi^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$, we define the *Cauchy index* of ψ by

$$\begin{aligned}\mathrm{ind}\,\psi &:= \sum_{l\in\Theta} \mathrm{ind}_l\,\psi + \sum_{x\in\Lambda_\psi\backslash\{\infty\}} \left(-\frac{1}{2} + \left\{\frac{1}{2} + \frac{1}{2\pi}\arg\frac{\psi(x+0)}{\psi(x-0)}\right\}\right) \\ &= \sum_{l\in\Theta} \mathrm{ind}_l\,\psi + \sum_{x\in\Lambda_\psi\backslash\{\infty\}} \left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg\frac{\psi(x+0)}{\psi(x-0)}\right\}\right).\end{aligned} \tag{5.2.27}$$

Considering $\arg\big(\psi(x+0)/\psi(x-0)\big) \in (-\pi,\pi)$, for all $x \in \Lambda_\psi \backslash \{\infty\}$, we have

$$\operatorname{ind}\psi = \sum_{l\in\Theta} \operatorname{ind}_l \psi + \frac{1}{2\pi} \sum_{x\in\Lambda_\psi\backslash\{\infty\}} \arg\frac{\psi(x+0)}{\psi(x-0)}. \tag{5.2.28}$$

The geometric meaning of the Cauchy index of ψ is therefore $(2\pi)^{-1}$ times the increment of the argument of z when z moves along the continuous curve $\psi^{\#}\big(\dot{\mathbb{R}},[0,1]\big)$ from $\psi(-\infty)$ to $\psi(+\infty)$. Consider now $\psi \in PC$ having countably many jumps. In this case, if $\psi^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$, we can also define the Cauchy index of ψ. For this purpose, we can uniformly approximate ψ by $\psi_n \in PC$ with finitely many jumps and such that $\psi_n^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$, with $\psi_n(\pm\infty) = \psi(\pm\infty)$. Then the Cauchy index of ψ is defined by

$$\operatorname{ind}\psi := \lim_{n\to+\infty} \operatorname{ind}\psi_n. \tag{5.2.29}$$

Considering the above definitions of winding number and Cauchy index, we have the following relation between the winding number and the Cauchy index of a piecewise continuous function:

$$\begin{aligned}
\operatorname{wind}\psi &= \operatorname{ind}\psi - \frac{1}{2} + \left\{\frac{1}{2} + \frac{1}{2\pi}\arg\frac{\psi(-\infty)}{\psi(+\infty)}\right\} \\
&= \operatorname{ind}\psi + \frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg\frac{\psi(-\infty)}{\psi(+\infty)}\right\} \\
&= \operatorname{ind}\psi + \frac{1}{2\pi}\arg\frac{\psi(-\infty)}{\psi(+\infty)},
\end{aligned} \tag{5.2.30}$$

where the last equality is valid if and only if $\arg\left(\psi(-\infty)/\psi(+\infty)\right) \in (-\pi,\pi)$.

Now having already defined the winding number of piecewise continuous functions, we are in position to present the definition of winding number of piecewise almost periodic functions. Recall from Proposition 2.3.1 that, for $\phi \in \mathcal{G}PAP$, ϕ can be represented as

$$\phi = \varphi\,\psi, \tag{5.2.31}$$

where $\varphi \in \mathcal{G}SAP$ and $\psi \in \mathcal{G}PC$ satisfying $\psi(-\infty) = \psi(+\infty) = 1$. Considering this representation of ϕ, A. Böttcher, Yu. I. Karlovich and I. M. Spitkovsky presented in [12] the following definition of the winding number of piecewise almost periodic functions. If

$\phi \in \mathcal{G}PAP$ is such that $\kappa_l(\phi) = \kappa_r(\phi) = 0$, $0 \notin [\mathbf{d}_l(\phi), \mathbf{d}_r(\phi)]$, and $\psi^{\#}(x, \mu) \neq 0$ for all $(x, \mu) \in \dot{\mathbb{R}} \times [0, 1]$, then the winding number of ϕ is defined by

$$\operatorname{wind} \phi := \operatorname{wind} \varphi + \operatorname{wind} \psi. \tag{5.2.32}$$

After having introduced a definition for the winding number of piecewise almost periodic functions, A. Böttcher, Yu. I. Karlovich and I. M. Spitkovsky presented in [12] a generalization of the Fredholm index formula obtained by D. Sarason, i.e., a Fredholm index formula for Wiener-Hopf operators with piecewise almost periodic Fourier symbols upon the winding number of the corresponding Fourier symbols.

Theorem 5.2.1. (cf. [12, Theorem 3.16]) *If* $\phi \in \mathcal{G}PAP$, $\kappa_l(\phi) = \kappa_r(\phi) = 0$ *and*

$$0 \notin [\mathbf{d}_l(\phi), \mathbf{d}_r(\phi)] \cup \bigcup_{x \in \mathbb{R}} \Big[\phi(x-0), \phi(x+0)\Big], \tag{5.2.33}$$

then

$$\operatorname{Ind} W_\phi = -\operatorname{wind} \phi. \tag{5.2.34}$$

As we will see later, we obtain a similar result in the framework of Wiener-Hopf-Hankel operators. To accomplish this result, and in the same way that we had to generalize the definition of winding number of semi-almost periodic functions in the context of Wiener-Hopf operators to the context of Wiener-Hopf-Hankel operators, here we also have to generalize the definition of winding number of piecewise almost periodic functions in the same sense, since from Theorem 5.1.2 we have that $(W \pm H)_\phi$ are Fredholm operators if $\phi \in \mathcal{G}PAP$ is such that $\kappa_l(\phi) + \kappa_r(\phi) = 0$ and

$$0 \notin \left[\frac{\mathbf{d}_l(\phi)}{\mathbf{d}_r(\phi)}, \frac{\mathbf{d}_r(\phi)}{\mathbf{d}_l(\phi)}\right] \cup \bigcup_{x \in \mathbb{R}} \Big[(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)\Big]. \tag{5.2.35}$$

Notice that in Definition 4.3.3 we introduced an appropriated definition of winding number of semi-almost periodic functions in the context of Wiener-Hopf-Hankel operators. Such generalization of the winding number of a semi-almost periodic function gives the answer in the generalization of the existent definition of winding number of a piecewise almost periodic function, as we can see in the following definition.

Definition 5.2.2. *For $\phi \in \mathcal{G}PAP$, consider that ϕ is represented as in* (2.3.38)*, $\phi = \varphi\psi$, and with $\psi^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$. If $\kappa_l(\phi) = \kappa_r(\phi) = 0$ and $0 \notin [\mathbf{d}_l(\phi), \mathbf{d}_r(\phi)]$ or if $\kappa_l(\phi) + \kappa_r(\phi) = 0$, $\Re e(\mathbf{d}(\phi_l)/\mathbf{d}(\phi_r)) \neq 0$ and $0 \notin [\mathbf{d}_l(\phi), \mathbf{d}_r(\phi)]$, then the winding number of ϕ is defined by*

$$\operatorname{wind}\phi := \operatorname{wind}\varphi + \operatorname{wind}\psi. \tag{5.2.36}$$

Having the winding number notion in the framework of Definition 5.2.2, we will now proceed and present a Fredholm index formula for the sum of Fredholm Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators with piecewise almost periodic Fourier symbols.

Theorem 5.2.3. *If $\phi \in \mathcal{G}PAP$, $\kappa_l(\phi) + \kappa_r(\phi) = 0$ and*

$$0 \notin \left[\frac{\mathbf{d}_l(\phi)}{\mathbf{d}_r(\phi)}, \frac{\mathbf{d}_r(\phi)}{\mathbf{d}_l(\phi)}\right] \cup \bigcup_{x\in\mathbb{R}} \left[(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)\right], \tag{5.2.37}$$

then

$$\operatorname{Ind}(W+H)_\phi + \operatorname{Ind}(W-H)_\phi = -\operatorname{wind}\left(\phi\widetilde{\phi^{-1}}\right). \tag{5.2.38}$$

Moreover, if $\Re e\left(\mathbf{d}_l(\phi)/\mathbf{d}_r(\phi)\right) > 0$ and:

(a) *ϕ does not have symmetric discontinuities, then*

$$\operatorname{Ind}(W+H)_\phi + \operatorname{Ind}(W-H)_\phi = -2\operatorname{wind}\phi\ ; \tag{5.2.39}$$

(b) *ϕ has symmetric discontinuities and ϕ is continuous at 0, then*

$$\operatorname{Ind}(W+H)_\phi + \operatorname{Ind}(W-H)_\phi = -2\left(\operatorname{wind}\varphi + \operatorname{wind}\varrho_\psi\right), \tag{5.2.40}$$

considering $\phi = \varphi\psi$ (cf. the representation (2.3.38) *of ϕ), and*

$$\varrho_\psi(x) := \begin{cases} \psi(0)\left(\psi\widetilde{\psi^{-1}}\right)(x) & \text{if } x \leq 0 \\ \psi(0) & \text{if } x > 0 \end{cases}; \tag{5.2.41}$$

(c) *ϕ is discontinuous at 0, then*

$$\begin{aligned}&\mathrm{Ind}\,(W+H)_\phi + \mathrm{Ind}\,(W-H)_\phi \\ &= -2\left(\mathrm{wind}\,\varphi + \mathrm{ind}\left(\left(\psi\widetilde{\psi^{-1}}\right)_-\right)\right) - \frac{1}{2\pi}\arg\left(\left(\frac{\psi(0+0)}{\psi(0-0)}\right)^2\right), \end{aligned} \tag{5.2.42}$$

considering $\phi = \varphi\psi$ (in the sense of the representation (2.3.38)),

$$\left(\psi\widetilde{\psi^{-1}}\right)_-(x) := \begin{cases} \left(\psi\widetilde{\psi^{-1}}\right)(x) & \text{if } x \leq 0 \\ \left(\psi\widetilde{\psi^{-1}}\right)(0-0) & \text{if } x > 0 \end{cases} \tag{5.2.43}$$

and

$$\arg\left(\left(\frac{\psi(0+0)}{\psi(0-0)}\right)^2\right) \in (-\pi, \pi). \tag{5.2.44}$$

Proof. Under the hypothesis, Theorems 5.1.1 and 5.1.2 ensure that $W_{\phi\widetilde{\phi^{-1}}}$, $(W+H)_\phi$ and $(W-H)_\phi$ are all Fredholm operators. Recalling now that $(W+H)_\phi$ is Δ–related after extension with $W_{\phi\widetilde{\phi^{-1}}}$ (cf. Lemma 1.3.7), it holds that

$$\mathrm{Ind}\,W_{\phi\widetilde{\phi^{-1}}} = \mathrm{Ind}\,(W+H)_\phi + \mathrm{Ind}\,\mathcal{T}_\phi. \tag{5.2.45}$$

In Proposition 1.3.9 it was proved that the operator $\mathcal{T}_\phi$ is equivalent after extension to the Wiener-Hopf minus Hankel operator $(W-H)_\phi$. Therefore, we have

$$\mathrm{Ind}\,\mathcal{T}_\phi = \mathrm{Ind}\,(W-H)_\phi. \tag{5.2.46}$$

Combining the two last identities, it results that

$$\mathrm{Ind}\,(W+H)_\phi + \mathrm{Ind}\,(W-H)_\phi = \mathrm{Ind}\,W_{\phi\widetilde{\phi^{-1}}}. \tag{5.2.47}$$

From the Fredholm index formula for Wiener-Hopf operators with *PAP* Fourier symbols presented in Theorem 5.2.1, we have

$$\mathrm{Ind}\,W_{\phi\widetilde{\phi^{-1}}} = -\,\mathrm{wind}\left(\phi\widetilde{\phi^{-1}}\right). \tag{5.2.48}$$

Thus, combining (5.2.47) and (5.2.48), it follows

$$\mathrm{Ind}\,(W+H)_\phi + \mathrm{Ind}\,(W-H)_\phi = -\mathrm{wind}\left(\phi\widetilde{\phi^{-1}}\right). \tag{5.2.49}$$

Having in mind the representation (2.3.38) of ϕ (i.e., $\phi = \varphi\psi$ with φ and ψ in the indicated classes), it holds

$$\phi\widetilde{\phi^{-1}} = \varphi\widetilde{\varphi^{-1}}\,\psi\widetilde{\psi^{-1}} \tag{5.2.50}$$

where $\varphi\widetilde{\varphi^{-1}} \in \mathcal{G}SAP$, and $\psi\widetilde{\psi^{-1}} \in \mathcal{G}PC$ is such that $\left(\psi\widetilde{\psi^{-1}}\right)(-\infty) = \left(\psi\widetilde{\psi^{-1}}\right)(+\infty) = 1$. According to Definition 5.2.2, one gets

$$\operatorname{wind}\left(\phi\widetilde{\phi^{-1}}\right) = \operatorname{wind}\left(\varphi\widetilde{\varphi^{-1}}\right) + \operatorname{wind}\left(\psi\widetilde{\psi^{-1}}\right). \tag{5.2.51}$$

In this case, taking into account that $\left(\varphi\widetilde{\varphi^{-1}}\right)_l = \left(\phi\widetilde{\phi^{-1}}\right)_l$ and $\left(\varphi\widetilde{\varphi^{-1}}\right)_r = \left(\phi\widetilde{\phi^{-1}}\right)_r$, it follows that $\operatorname{wind}\left(\varphi\widetilde{\varphi^{-1}}\right)$ is well defined since

$$\kappa\Big(\left(\varphi\widetilde{\varphi^{-1}}\right)_l\Big) = \kappa\Big(\left(\varphi\widetilde{\varphi^{-1}}\right)_r\Big) = \kappa_l(\phi) + \kappa_r(\phi) = 0 \tag{5.2.52}$$

and

$$0 \notin \left[\mathbf{d}\Big(\left(\varphi\widetilde{\varphi^{-1}}\right)_l\Big), \mathbf{d}\Big(\left(\varphi\widetilde{\varphi^{-1}}\right)_r\Big)\right] = \left[\frac{\mathbf{d}_l(\phi)}{\mathbf{d}_r(\phi)}, \frac{\mathbf{d}_r(\phi)}{\mathbf{d}_l(\phi)}\right]. \tag{5.2.53}$$

From the proof of Theorem 4.3.6 (see (4.3.144)), we already know how to relate $\operatorname{wind}\varphi$ with $\operatorname{wind}(\varphi\widetilde{\varphi^{-1}})$ when $\Re e(\mathbf{d}(\varphi_l)/\mathbf{d}(\varphi_r)) > 0$, i.e., when $\Re e(\mathbf{d}_l(\phi)/\mathbf{d}_r(\phi)) > 0$:

$$\operatorname{wind}\left(\varphi\widetilde{\varphi^{-1}}\right) = 2\operatorname{wind}\varphi. \tag{5.2.54}$$

Let us now look for an identity involving $\operatorname{wind}\left(\psi\widetilde{\psi^{-1}}\right)$ and $\operatorname{wind}\psi$. For this purpose, we will analyze the functions $\psi^{\#}, \left(\widetilde{\psi^{-1}}\right)^{\#}, \left(\psi\widetilde{\psi^{-1}}\right)^{\#} : \dot{\mathbb{R}} \times [0,1] \to \mathbb{C}$. For all $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$, we have

$$\begin{aligned}\left(\psi\widetilde{\psi^{-1}}\right)^{\#}&(x,\mu) - \psi^{\#}(x,\mu)\left(\widetilde{\psi^{-1}}\right)^{\#}(x,\mu)\\ &= (1-\mu)\mu\left(\psi(x+0) - \psi(x-0)\right)\left(\widetilde{\psi^{-1}}(x+0) - \widetilde{\psi^{-1}}(x-0)\right).\end{aligned} \tag{5.2.55}$$

We will now consider three different situations: (1) ψ has no symmetric discontinuities; (2) ψ has symmetric discontinuities and ψ is continuous at 0; and finally, (3) ψ is discontinuous at 0. Since the discontinuities of ϕ arise from the factor ψ (in the factorization $\phi = \varphi\psi$), we are in fact facing the following cases: (a) ϕ has no symmetric discontinuities; (b) ϕ has symmetric discontinuities and ϕ is continuous at 0; and, (c) ϕ is discontinuous at 0.

Case 1: Since in this case ψ and $\widetilde{\psi^{-1}}$ have no common discontinuities, we conclude from (5.2.55) that

$$\left(\psi\widetilde{\psi^{-1}}\right)^{\#}(x,\mu) = \psi^{\#}(x,\mu)\left(\widetilde{\psi^{-1}}\right)^{\#}(x,\mu) \tag{5.2.56}$$

for all $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$. Thus, observing that $\psi^{\#}$ and $\left(\widetilde{\psi^{-1}}\right)^{\#}$ are closed continuous curves away from zero, the identity holds

$$\operatorname{wind}\left(\psi\widetilde{\psi^{-1}}\right)^{\#} = \operatorname{wind}\psi^{\#} + \operatorname{wind}\left(\widetilde{\psi^{-1}}\right)^{\#}, \tag{5.2.57}$$

(cf. e.g. [14, §2.41]). Furthermore, $\operatorname{wind}\psi^{\#} = \operatorname{ind}\psi^{\#}$ and $\operatorname{wind}\left(\widetilde{\psi^{-1}}\right)^{\#} = \operatorname{ind}\left(\widetilde{\psi^{-1}}\right)^{\#}$. Computing now the Cauchy index of $\left(\widetilde{\psi^{-1}}\right)^{\#}$, one gets

$$\begin{aligned}
\operatorname{ind}\left(\widetilde{\psi^{-1}}\right)^{\#} &= \frac{1}{2\pi}\Big[\Big(\arg\left(\widetilde{\psi^{-1}}\right)^{\#}\Big)(+\infty) - \Big(\arg\left(\widetilde{\psi^{-1}}\right)^{\#}\Big)(-\infty)\Big] \\
&= \frac{1}{2\pi}\big[(\arg\widetilde{\psi^{-1}})(+\infty) - (\arg\widetilde{\psi^{-1}})(-\infty)\big] \\
&= \frac{1}{2\pi}\big[-(\arg\psi)(-\infty) + (\arg\psi)(+\infty)\big] \\
&= \frac{1}{2\pi}\Big[\Big(\arg\psi^{\#}\Big)(+\infty) - \Big(\arg\psi^{\#}\Big)(-\infty)\Big] \\
&= \operatorname{ind}\psi^{\#},
\end{aligned} \tag{5.2.58}$$

i.e.,

$$\operatorname{wind}\left(\widetilde{\psi^{-1}}\right)^{\#} = \operatorname{wind}\psi^{\#}. \tag{5.2.59}$$

From (5.2.57) and (5.2.59), it follows

$$\operatorname{wind}\left(\psi\widetilde{\psi^{-1}}\right)^{\#} = 2\operatorname{wind}\psi^{\#}. \tag{5.2.60}$$

Applying the definition of winding number for piecewise continuous functions, we obtain

$$\operatorname{wind}\left(\psi\widetilde{\psi^{-1}}\right) = 2\operatorname{wind}\psi. \tag{5.2.61}$$

From (5.2.51), (5.2.54), (5.2.61), and Definition 5.2.2, we have

$$\operatorname{wind}\left(\phi\widetilde{\phi^{-1}}\right) = 2\operatorname{wind}\varphi + 2\operatorname{wind}\psi = 2\operatorname{wind}\phi. \tag{5.2.62}$$

According to (5.2.49), it follows that

$$\operatorname{Ind}(W+H)_\phi + \operatorname{Ind}(W-H)_\phi = -2\,\operatorname{wind}\phi. \tag{5.2.63}$$

Case 2: In this case, we may write

$$\psi = \psi_-\psi_+ \tag{5.2.64}$$

in such a way that $\psi_-, \psi_+ \in PC$ do not have common discontinuities. For that, consider

$$\psi_-(x) = \begin{cases} \psi(x) & \text{if } x \le 0 \\ \psi(0) & \text{if } x > 0 \end{cases} \qquad \text{and} \qquad \psi_+(x) = \begin{cases} 1 & \text{if } x \le 0 \\ \dfrac{\psi(x)}{\psi(0)} & \text{if } x > 0 \end{cases}, \tag{5.2.65}$$

where ψ_- has only discontinuities in $\mathbb{R}_-$ and ψ_+ has only discontinuities in $\mathbb{R}_+$. From (5.2.64), it follows

$$\psi\widetilde{\psi^{-1}} = \psi_-\psi_+\widetilde{\psi_+^{-1}}\widetilde{\psi_-^{-1}} = \left(\psi_-\widetilde{\psi_+^{-1}}\right)\left(\psi_+\widetilde{\psi_-^{-1}}\right). \tag{5.2.66}$$

Computing $\psi_-\widetilde{\psi_+^{-1}}$ and $\psi_+\widetilde{\psi_-^{-1}}$, one gets

$$\left(\psi_-\widetilde{\psi_+^{-1}}\right)(x) = \begin{cases} \psi(0)\dfrac{\psi(x)}{\psi(-x)} & \text{if } x \le 0 \\ \psi(0) & \text{if } x > 0 \end{cases}, \tag{5.2.67}$$

$$\left(\psi_+\widetilde{\psi_-^{-1}}\right)(x) = \begin{cases} \dfrac{1}{\psi(0)} & \text{if } x \le 0 \\ \dfrac{\psi(x)}{\psi(0)\psi(-x)} & \text{if } x > 0 \end{cases}. \tag{5.2.68}$$

From here, we see that $\psi_-\widetilde{\psi_+^{-1}}$ is a function with discontinuities in $\mathbb{R}_-$ while $\psi_+\widetilde{\psi_-^{-1}}$ is a function with discontinuities in $\mathbb{R}_+$. Putting $\varrho_\psi = \psi_-\widetilde{\psi_+^{-1}}$, it follows that $\psi_+\widetilde{\psi_-^{-1}} = \widetilde{\varrho_\psi^{-1}}$. Rewriting (5.2.66) in terms of ϱ_ψ and $\widetilde{\varrho_\psi^{-1}}$, we obtain

$$\psi\widetilde{\psi^{-1}} = \varrho_\psi\widetilde{\varrho_\psi^{-1}}, \tag{5.2.69}$$

where ϱ_ψ and $\widetilde{\varrho_\psi^{-1}}$ do not have common discontinuities. Additionally, from the hypothesis

$$0 \notin \bigcup_{x\in\mathbb{R}} \left[(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)\right], \tag{5.2.70}$$

it follows that

$$0 \notin \bigcup_{x \in \mathbb{R}} \left[(\psi\widetilde{\psi^{-1}})(x-0), (\psi\widetilde{\psi^{-1}})(x+0) \right], \tag{5.2.71}$$

i.e., $\left(\psi\widetilde{\psi^{-1}}\right)^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$. Therefore, we have $\varrho_{\psi}^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$ and $\widetilde{\varrho_{\psi}^{-1}}^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$, which means that $\operatorname{wind} \varrho_{\psi}$ and $\operatorname{wind} \widetilde{\varrho_{\psi}^{-1}}$ are well defined. Following now the same reasoning as in Case 1, we obtain

$$\operatorname{wind}\left(\psi\widetilde{\psi^{-1}}\right) = 2 \operatorname{wind} \varrho_{\psi}. \tag{5.2.72}$$

Finally, from (5.2.51), (5.2.54), (5.2.72), we have

$$\operatorname{wind}\left(\phi\widetilde{\phi^{-1}}\right) = 2 \left(\operatorname{wind} \varphi + \operatorname{wind} \varrho_{\psi}\right), \tag{5.2.73}$$

which, by (5.2.49), yields that

$$\operatorname{Ind}(W+H)_{\phi} + \operatorname{Ind}(W-H)_{\phi} = -2\left(\operatorname{wind} \varphi + \operatorname{wind} \varrho_{\psi}\right). \tag{5.2.74}$$

Case 3: Since ψ is discontinuous at 0, we may identify ψ with

$$\psi(x) = \begin{cases} \psi_1(x) & \text{if } x \leq 0 \\ \psi_2(x) & \text{if } x > 0 \end{cases}, \tag{5.2.75}$$

where $\psi_1, \psi_2 \in \mathcal{G}PC$ are such that $\psi_1(0) \neq \psi_2(0+0)$ and $\psi_1(-\infty) = \psi_2(+\infty) = 1$ (recall that from Proposition 2.3.1 we have $\psi(-\infty) = \psi(+\infty) = 1$). Thus, it follows that $\psi\widetilde{\psi^{-1}}$ may be written in the form

$$\left(\psi\widetilde{\psi^{-1}}\right)(x) = \begin{cases} \left(\psi_1\widetilde{\psi_2^{-1}}\right)(x) & \text{if } x \leq 0 \\ \left(\widetilde{\psi_1^{-1}}\psi_2\right)(x) & \text{if } x > 0 \end{cases}. \tag{5.2.76}$$

Due to the equality $\left(\psi\widetilde{\psi^{-1}}\right)(-\infty) = \left(\psi\widetilde{\psi^{-1}}\right)(+\infty)$, and according to (5.2.30), it follows that

$$\operatorname{wind}\left(\psi\widetilde{\psi^{-1}}\right) = \operatorname{ind}\left(\psi\widetilde{\psi^{-1}}\right). \tag{5.2.77}$$

In order to compute the Cauchy index of $\psi\widetilde{\psi^{-1}}$, let us assume (without lost of generality) that $\psi\widetilde{\psi^{-1}}$ has finitely many jumps. It is clear that the discontinuities of $\psi\widetilde{\psi^{-1}}$ are symmetric. Note however that ψ admits a discontinuity at 0 but $\psi\widetilde{\psi^{-1}}$ is not always discontinuous at 0. Hence

$$\Lambda_{\psi\widetilde{\psi^{-1}}} = \{-x_n, \ldots, -x_1, x_0, x_1, \ldots, x_n\} \tag{5.2.78}$$

or

$$\Lambda_{\psi\widetilde{\psi^{-1}}} = \{-x_n, \ldots, -x_1, x_1, \ldots, x_n\}, \tag{5.2.79}$$

with $n \in \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$, $x_j \in \mathbb{R}_+$ (for all $j = 1, \ldots, n$), and $x_0 = 0$. Let us first consider the case $\Lambda_{\psi\widetilde{\psi^{-1}}} = \{-x_n, \ldots, -x_1, x_0, x_1, \ldots, x_n\}$. Taking into account the definition of Cauchy index of a piecewise continuous function, we get

$$\begin{aligned} \operatorname{ind}\left(\psi\widetilde{\psi^{-1}}\right) &= \operatorname{ind}_{]-\infty,-x_n[}\left(\psi\widetilde{\psi^{-1}}\right) + \sum_{j=1}^{n} \operatorname{ind}_{]-x_j,-x_{j-1}[}\left(\psi\widetilde{\psi^{-1}}\right) \\ &+ \sum_{j=1}^{n} \operatorname{ind}_{]x_{j-1},x_j[}\left(\psi\widetilde{\psi^{-1}}\right) + \operatorname{ind}_{]x_n,+\infty[}\left(\psi\widetilde{\psi^{-1}}\right) \\ &+ \sum_{x\in\Lambda_{\psi\widetilde{\psi^{-1}}}} \frac{1}{2\pi} \arg \frac{\left(\psi\widetilde{\psi^{-1}}\right)(x+0)}{\left(\psi\widetilde{\psi^{-1}}\right)(x-0)}, \end{aligned} \tag{5.2.80}$$

i.e.,

$$\begin{aligned} \operatorname{ind}\left(\psi\widetilde{\psi^{-1}}\right) &= \operatorname{ind}_{]-\infty,-x_n[}\left(\psi\widetilde{\psi^{-1}}\right) + \sum_{j=1}^{n} \operatorname{ind}_{]-x_j,-x_{j-1}[}\left(\psi\widetilde{\psi^{-1}}\right) \\ &+ \sum_{x\in\Lambda_{\psi\widetilde{\psi^{-1}}}\cap\mathbb{R}_-} \frac{1}{2\pi} \arg \frac{\left(\psi\widetilde{\psi^{-1}}\right)(x+0)}{\left(\psi\widetilde{\psi^{-1}}\right)(x-0)} \\ &+ \frac{1}{2\pi} \arg \frac{\left(\psi\widetilde{\psi^{-1}}\right)(0+0)}{\left(\psi\widetilde{\psi^{-1}}\right)(0-0)} \\ &+ \sum_{x\in\Lambda_{\psi\widetilde{\psi^{-1}}}\cap\mathbb{R}_+} \frac{1}{2\pi} \arg \frac{\left(\psi\widetilde{\psi^{-1}}\right)(x+0)}{\left(\psi\widetilde{\psi^{-1}}\right)(x-0)} \\ &+ \sum_{j=1}^{n} \operatorname{ind}_{]x_{j-1},x_j[}\left(\psi\widetilde{\psi^{-1}}\right) + \operatorname{ind}_{]x_n,+\infty[}\left(\psi\widetilde{\psi^{-1}}\right), \end{aligned} \tag{5.2.81}$$

where $\arg\left(\left(\psi\widetilde{\psi^{-1}}\right)(x+0)/\left(\psi\widetilde{\psi^{-1}}\right)(x-0)\right) \in (-\pi,\pi)$, for all $x \in \Lambda_{\psi\widetilde{\psi^{-1}}}$. Using now (5.2.76), it follows that

$$\begin{aligned}
\operatorname{ind}\left(\psi\widetilde{\psi^{-1}}\right) &= \operatorname{ind}_{]-\infty,-x_n[}\left(\psi_1\widetilde{\psi_2^{-1}}\right) + \sum_{j=1}^{n}\operatorname{ind}_{]-x_j,-x_{j-1}[}\left(\psi_1\widetilde{\psi_2^{-1}}\right)\\
&\quad + \sum_{x\in\Lambda_{\psi\widetilde{\psi^{-1}}}\cap\mathbb{R}_-}\frac{1}{2\pi}\arg\frac{\left(\psi_1\widetilde{\psi_2^{-1}}\right)(x+0)}{\left(\psi_1\widetilde{\psi_2^{-1}}\right)(x-0)}\\
&\quad + \frac{1}{2\pi}\arg\frac{\left(\psi\widetilde{\psi^{-1}}\right)(0+0)}{\left(\psi\widetilde{\psi^{-1}}\right)(0-0)}\\
&\quad + \sum_{x\in\Lambda_{\psi\widetilde{\psi^{-1}}}\cap\mathbb{R}_+}\frac{1}{2\pi}\arg\frac{\left(\widetilde{\psi_1^{-1}}\psi_2\right)(x+0)}{\left(\widetilde{\psi_1^{-1}}\psi_2\right)(x-0)}\\
&\quad + \sum_{j=1}^{n}\operatorname{ind}_{]x_{j-1},x_j[}\left(\widetilde{\psi_1^{-1}}\psi_2\right) + \operatorname{ind}_{]x_n,+\infty[}\left(\widetilde{\psi_1^{-1}}\psi_2\right).
\end{aligned}\tag{5.2.82}$$

Noticing that $\left(\widetilde{\psi_1^{-1}}\psi_2\right)(x) = \widetilde{\left(\psi_1\widetilde{\psi_2^{-1}}\right)^{-1}}(x)$, for all $x>0$, we have

$$\left(\arg\left(\widetilde{\psi_1^{-1}}\psi_2\right)\right)(x) = -\left(\arg\left(\psi_1\widetilde{\psi_2^{-1}}\right)\right)(-x)\tag{5.2.83}$$

for all $x>0$. Therefore, we obtain the following identities:

$$\begin{aligned}
\operatorname{ind}_{]x_n,+\infty[}\left(\widetilde{\psi_1^{-1}}\psi_2\right) &= \frac{1}{2\pi}\left(\left(\arg\left(\widetilde{\psi_1^{-1}}\psi_2\right)\right)(+\infty) - \left(\arg\left(\widetilde{\psi_1^{-1}}\psi_2\right)\right)(x_n)\right)\\
&= \frac{1}{2\pi}\left(\left(\arg\left(\psi_1\widetilde{\psi_2^{-1}}\right)\right)(-x_n) - \left(\arg\left(\psi_1\widetilde{\psi_2^{-1}}\right)\right)(-\infty)\right)\\
&= \operatorname{ind}_{]-\infty,-x_n[}\left(\psi_1\widetilde{\psi_2^{-1}}\right),
\end{aligned}\tag{5.2.84}$$

$$\begin{aligned}
\operatorname{ind}_{]x_{j-1},x_j[}\left(\widetilde{\psi_1^{-1}}\psi_2\right) &= \frac{1}{2\pi}\left(\left(\arg\left(\widetilde{\psi_1^{-1}}\psi_2\right)\right)(x_j) - \left(\arg\left(\widetilde{\psi_1^{-1}}\psi_2\right)\right)(x_{j-1})\right)\\
&= \frac{1}{2\pi}\left(\left(\arg\left(\psi_1\widetilde{\psi_2^{-1}}\right)\right)(-x_{j-1}) - \left(\arg\left(\psi_1\widetilde{\psi_2^{-1}}\right)\right)(-x_j)\right)\\
&= \operatorname{ind}_{]-x_j,-x_{j-1}[}\left(\psi_1\widetilde{\psi_2^{-1}}\right),
\end{aligned}\tag{5.2.85}$$

for all $j = 1, \dots, n$. Moreover, it holds

$$\frac{\left(\widetilde{\psi_1^{-1}\psi_2}\right)(x+0)}{\left(\widetilde{\psi_1^{-1}\psi_2}\right)(x-0)} = \frac{\left(\psi_1^{-1}\widetilde{\psi_2}\right)(-x-0)}{\left(\psi_1^{-1}\widetilde{\psi_2}\right)(-x+0)} = \frac{\left(\psi_1\widetilde{\psi_2^{-1}}\right)(-x+0)}{\left(\psi_1\widetilde{\psi_2^{-1}}\right)(-x-0)}\,, \tag{5.2.86}$$

for all $x > 0$. Inserting (5.2.84), (5.2.85) and (5.2.86) in (5.2.82), we obtain

$$\begin{aligned} \operatorname{ind}\left(\psi\widetilde{\psi^{-1}}\right) = 2 \Bigg(& \operatorname{ind}_{]-\infty,-x_n[}\left(\psi_1\widetilde{\psi_2^{-1}}\right) + \sum_{j=1}^{n} \operatorname{ind}_{]-x_j,-x_{j-1}[}\left(\psi_1\widetilde{\psi_2^{-1}}\right) \\ & + \sum_{x\in\Lambda_{\psi\widetilde{\psi^{-1}}}\cap\mathbb{R}_-} \frac{1}{2\pi}\arg\frac{\left(\psi_1\widetilde{\psi_2^{-1}}\right)(x+0)}{\left(\psi_1\widetilde{\psi_2^{-1}}\right)(x-0)}\Bigg) + \frac{1}{2\pi}\arg\frac{\left(\psi\widetilde{\psi^{-1}}\right)(0+0)}{\left(\psi\widetilde{\psi^{-1}}\right)(0-0)}\,, \end{aligned} \tag{5.2.87}$$

where $\arg\left(\left(\psi_1\widetilde{\psi_2^{-1}}\right)(x+0)\Big/\left(\psi_1\widetilde{\psi_2^{-1}}\right)(x-0)\right) \in (-\pi,\pi)$ for all $x \in \Lambda_{\psi\widetilde{\psi^{-1}}} \cap \mathbb{R}_-$ and $\arg\left(\left(\psi\widetilde{\psi^{-1}}\right)(0+0)/\left(\psi\widetilde{\psi^{-1}}\right)(0-0)\right) \in (-\pi,\pi)$. Considering the function $\left(\psi\widetilde{\psi^{-1}}\right)_-$ given by

$$\left(\psi\widetilde{\psi^{-1}}\right)_-(x) = \begin{cases} \left(\psi\widetilde{\psi^{-1}}\right)(x) & \text{if } x \le 0 \\ \left(\psi\widetilde{\psi^{-1}}\right)(0-0) & \text{if } x > 0 \end{cases} \tag{5.2.88}$$

$$= \begin{cases} \left(\psi_1\widetilde{\psi_2^{-1}}\right)(x) & \text{if } x \le 0 \\ \left(\psi_1\widetilde{\psi_2^{-1}}\right)(0-0) & \text{if } x > 0 \end{cases}, \tag{5.2.89}$$

we can represent the Cauchy index of $\psi\widetilde{\psi^{-1}}$ in the form

$$\operatorname{ind}\left(\psi\widetilde{\psi^{-1}}\right) = 2\operatorname{ind}\left(\left(\psi\widetilde{\psi^{-1}}\right)_-\right) + \frac{1}{2\pi}\arg\frac{\left(\psi\widetilde{\psi^{-1}}\right)(0+0)}{\left(\psi\widetilde{\psi^{-1}}\right)(0-0)}\,, \tag{5.2.90}$$

considering $\arg\left(\left(\psi\widetilde{\psi^{-1}}\right)(0+0)/\left(\psi\widetilde{\psi^{-1}}\right)(0-0)\right) \in (-\pi,\pi)$. Since

$$\frac{\left(\psi\widetilde{\psi^{-1}}\right)(0+0)}{\left(\psi\widetilde{\psi^{-1}}\right)(0-0)} = \left(\frac{\psi(0+0)}{\psi(0-0)}\right)^2, \tag{5.2.91}$$

it holds

$$\operatorname{ind}\left(\psi\widetilde{\psi^{-1}}\right) = 2\operatorname{ind}\left(\left(\psi\widetilde{\psi^{-1}}\right)_-\right) + \frac{1}{2\pi}\arg\left(\left(\frac{\psi(0+0)}{\psi(0-0)}\right)^2\right), \tag{5.2.92}$$

for $\arg\left(\left(\psi(0+0)/\psi(0-0)\right)^2\right) \in (-\pi,\pi)$. In the case where $\psi\widetilde{\psi^{-1}}$ is not discontinuous at 0, i.e., in the case where $\Lambda_{\psi\widetilde{\psi^{-1}}} = \{-x_n,\ldots,-x_1,x_1,\ldots,x_n\}$, (5.2.90) also holds, and so we have in this case

$$\operatorname{ind}\left(\psi\widetilde{\psi^{-1}}\right) = 2\operatorname{ind}\left(\left(\psi\widetilde{\psi^{-1}}\right)_-\right). \tag{5.2.93}$$

From (5.2.77) and (5.2.92), it follows then

$$\operatorname{wind}\left(\psi\widetilde{\psi^{-1}}\right) = 2\operatorname{ind}\left(\left(\psi\widetilde{\psi^{-1}}\right)_-\right) + \frac{1}{2\pi}\arg\left(\left(\frac{\psi(0+0)}{\psi(0-0)}\right)^2\right), \tag{5.2.94}$$

where $\arg\left(\left(\psi(0+0)/\psi(0-0)\right)^2\right) \in (-\pi,\pi)$. Taking into account (5.2.51), (5.2.54), (5.2.94), we obtain the formula for the winding number of $\phi\widetilde{\phi^{-1}}$:

$$\operatorname{wind}\left(\phi\widetilde{\phi^{-1}}\right) = 2\left(\operatorname{wind}\varphi + \operatorname{ind}\left(\left(\psi\widetilde{\psi^{-1}}\right)_-\right)\right) + \frac{1}{2\pi}\arg\left(\left(\frac{\psi(0+0)}{\psi(0-0)}\right)^2\right), \tag{5.2.95}$$

for $\arg\left(\left(\psi(0+0)/\psi(0-0)\right)^2\right) \in (-\pi,\pi)$. Finally, from (5.2.49) it holds

$$\begin{aligned}&\operatorname{Ind}(W+H)_\phi + \operatorname{Ind}(W-H)_\phi \\ &= -2\left(\operatorname{wind}\varphi + \operatorname{ind}\left(\left(\psi\widetilde{\psi^{-1}}\right)_-\right)\right) - \frac{1}{2\pi}\arg\left(\left(\frac{\psi(0+0)}{\psi(0-0)}\right)^2\right),\end{aligned} \tag{5.2.96}$$

considering $\arg\left(\left(\psi(0+0)/\psi(0-0)\right)^2\right) \in (-\pi,\pi)$. □

Remark 5.2.4. Although in the case where ψ does not have symmetric discontinuities we have $\operatorname{wind}(\psi\widetilde{\psi^{-1}}) = 2\operatorname{wind}\psi$ (and consequently $\operatorname{wind}(\phi\widetilde{\phi^{-1}}) = 2\operatorname{wind}\phi$, if $\Re e\,(\mathbf{d}_l(\phi)/\mathbf{d}_r(\phi)) > 0$), this is – in general – not true for ψ with symmetric discontinuities (as we will see in Example 5.2.7). In this sense, we need to find another way to compute $\operatorname{wind}(\psi\widetilde{\psi^{-1}})$ based on a winding number or a Cauchy index of a function related with ψ. The obtained formula in the case where ψ has symmetric discontinuities and ψ is continuous at 0, $\operatorname{wind}(\psi\widetilde{\psi^{-1}}) = 2\operatorname{wind}\varrho_\psi$, can also be applied when ψ does not have symmetric discontinuities. In the case where ψ is discontinuous at 0, we obtain a more general formula that also applies to the previous first and second cases. We would like also to mentioned that, using a similar reasoning, it is possible to compute $\operatorname{wind}(\psi\widetilde{\psi^{-1}})$ in terms of a winding number or a Cauchy index of a piecewise continuous function related with ψ and having discontinuities in $\mathbb{R}_+$. Consequently, if in the conditions of cases (b) and (c) of Theorem 5.2.3, it is possible to derive formulae for the sum of the Fredholm indices of Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators with piecewise almost periodic Fourier symbols in terms of piecewise continuous functions with discontinuities in $\mathbb{R}_+$.

Finally, we would like also to point out that although it is very tempting to write

$$\begin{aligned}\operatorname{Ind}(W{+}H)_\phi + \operatorname{Ind}(W{-}H)_\phi &= \operatorname{Ind} W_\phi + \operatorname{Ind} H_\phi + \operatorname{Ind} W_\phi - \operatorname{Ind} H_\phi \\ &= 2 \operatorname{Ind} W_\phi, \end{aligned} \tag{5.2.97}$$

this is not true (for $\phi \in \mathcal{G}PAP$). In the next subsection we will provide two structural examples related with this issue. Example 5.2.5 shows that the Fredholm property may even do not held for all the operators used in (5.2.97) while Example 5.2.6 presents the case of Fredholm operators $(W{+}H)_\phi$, $(W{-}H)_\phi$ and W_ϕ such that the sum of the Fredholm indices of $(W{+}H)_\phi$ and $(W{-}H)_\phi$ is not equal to two times the Fredholm index of W_ϕ. To construct an example of such kind, we observe that the case of Fredholm operators $(W{+}H)_\phi$, $(W{-}H)_\phi$ and W_ϕ only occurs when $\kappa_l(\phi) = \kappa_r(\phi) = 0$, and

$$0 \notin \Big[\mathbf{d}_l(\phi), \mathbf{d}_r(\phi)\Big] \cup \bigcup_{x\in\mathbb{R}} \Big[\phi(x-0), \phi(x+0)\Big], \tag{5.2.98}$$

$$0 \notin \left[\frac{\mathbf{d}_l(\phi)}{\mathbf{d}_r(\phi)}, \frac{\mathbf{d}_r(\phi)}{\mathbf{d}_l(\phi)}\right] \cup \bigcup_{x\in\mathbb{R}} \Big[(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)\Big]. \tag{5.2.99}$$

This conclusion follows straightforward if we combine Theorem 5.1.1 with Theorem 5.1.2.

5.2.2 Examples and invertibility

To end up this chapter, we present some examples that illustrate the observations made in Remark 5.2.4, as well as a condition for the invertibility of Wiener-Hopf-Hankel operators with piecewise almost periodic Fourier symbols.

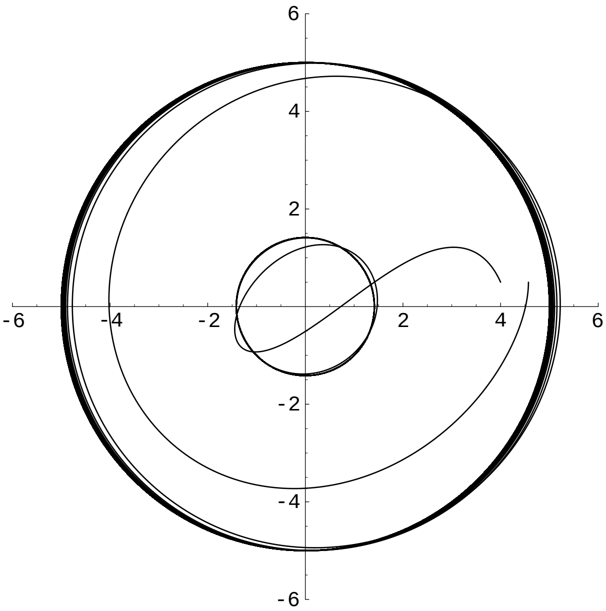


Figure 5.3: The range of $\phi(x)$ defined in (5.2.100) for x between -1000 and 1000.

Example 5.2.5. Consider the function ϕ (see Figure 5.3) given by

$$\phi(x) = (1 - u(x))\,(1 + i)\,e^{2ix} + u(x)\,5\,e^{-2ix} + \phi_0(x), \tag{5.2.100}$$

where

$$u(x) = \begin{cases} \frac{1}{2}\,e^{x} & \text{if } x \leq 0 \\ 1 - \frac{1}{2}\,e^{-x} & \text{if } x > 0 \end{cases} \quad \text{and} \quad \phi_0(x) = \begin{cases} e^{x} & \text{if } x \leq 0 \\ \operatorname{arc\,cot} x & \text{if } x > 0 \end{cases}. \tag{5.2.101}$$

From Theorem 5.1.1 and Theorem 5.1.2, we have that the Wiener-Hopf plus Hankel operator $(W{+}H)_\phi$ and the Wiener-Hopf minus Hankel operator $(W{-}H)_\phi$ are Fredholm operators while the Wiener-Hopf operator W_ϕ is not a Fredholm operator (since it is not normally solvable).

Example 5.2.6. Consider now the function ϕ (see Figure 5.4) given by

$$\phi(x) = (1 - u(x))\,i\,e^{e^{ix}} + u(x)\,(1 - i)\,e^{e^{ix}} + \phi_0(x), \tag{5.2.102}$$

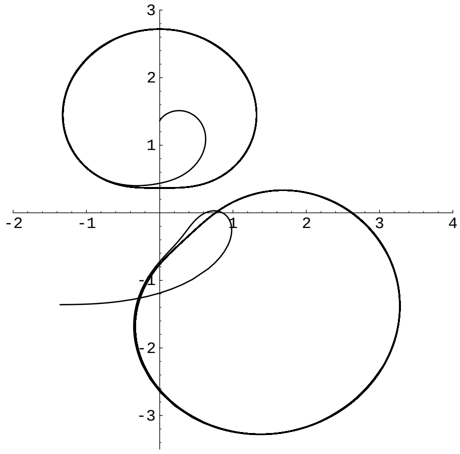


Figure 5.4: The range of $\phi(x)$ defined in (5.2.102) for x between -1000 and 1000.

where

$$u(x) = \frac{1}{2}(1+\tanh x) \qquad \text{and} \qquad \phi_0(x) = \begin{cases} \frac{1}{2}(i-1)e^{x+1} & \text{if } x \leq 0 \\ -\left(1+\frac{i}{2}\right)e^{-x+1} & \text{if } x > 0 \end{cases}. \tag{5.2.103}$$

Since $\phi \in \mathcal{G}PAP$, we may look for a representation of ϕ as (2.3.38). For instance, consider φ and ψ given by

$$\varphi(x) = (1-u(x))\, i\, e^{e^{ix}} + u(x)\,(1-i)\, e^{e^{ix}} \tag{5.2.104}$$

and

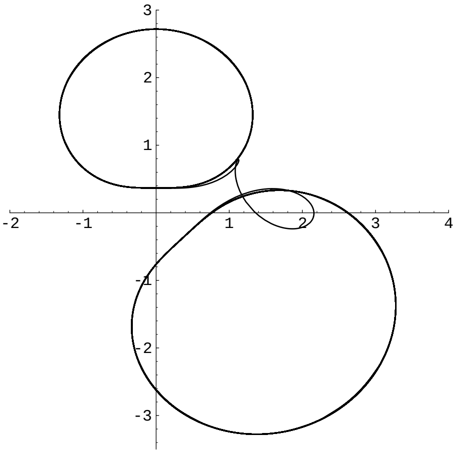
$$\psi(x) = 1 + (\phi_0\varphi^{-1})(x) = \begin{cases} 1 + \dfrac{(i-1)e^{x+1}}{2\varphi(x)} & \text{if } x \leq 0 \\ 1 - \dfrac{\left(1+\frac{i}{2}\right)e^{-x+1}}{\varphi(x)} & \text{if } x > 0 \end{cases}, \tag{5.2.105}$$

respectively. In this way, we get

$$\phi = \varphi\,\psi. \tag{5.2.106}$$

We see that $\varphi \in \mathcal{G}SAP$ and $\psi \in \mathcal{G}PC$ is such that $\psi(-\infty) = \psi(+\infty) = 1$. Computing the winding numbers of φ, ψ, $\varphi\widetilde{\varphi^{-1}}$ and $\psi\widetilde{\psi^{-1}}$ (see Figures 5.5, 5.6, 5.7 and 5.8), we have

$$\operatorname{wind}\varphi = 0, \quad \operatorname{wind}\left(\varphi\widetilde{\varphi^{-1}}\right) = -1, \quad \operatorname{wind}\psi = \operatorname{wind}\left(\psi\widetilde{\psi^{-1}}\right) = 1. \tag{5.2.107}$$



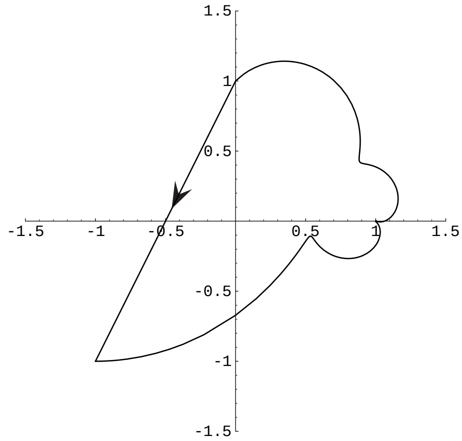
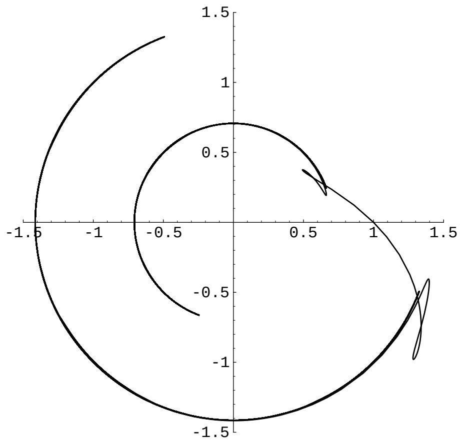


Figure 5.5: The range of $\varphi(x)$ defined in (5.2.104) for x between -500 and 500.

Figure 5.6: The oriented graph of $\psi^{\#}$, for ψ defined in (5.2.105).



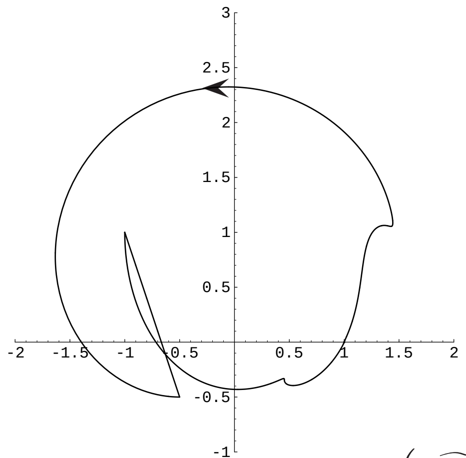


Figure 5.7: The range of $\left(\varphi\widetilde{\varphi^{-1}}\right)(x)$ for x between -1000 and 1000, where φ is defined in (5.2.104).

Figure 5.8: The oriented graph of $\left(\psi\widetilde{\psi^{-1}}\right)^{\#}$, for ψ defined in (5.2.105).

From Definition 5.2.2, we obtain

$$\operatorname{wind}\phi = 1, \quad \operatorname{wind}\left(\phi\widetilde{\phi^{-1}}\right) = 0. \tag{5.2.108}$$

Additionally, by the Fredholm index formulae presented in (5.2.34) and (5.2.38), it follows that

$$\operatorname{Ind} W_\phi = -1, \quad \operatorname{Ind}(W+H)_\phi + \operatorname{Ind}(W-H)_\phi = 0, \tag{5.2.109}$$

which shows that

$$\operatorname{Ind}(W+H)_\phi + \operatorname{Ind}(W-H)_\phi \neq 2\operatorname{Ind} W_\phi. \tag{5.2.110}$$

In order to exemplify the simplification for the formula of the winding number of $\psi\widetilde{\psi^{-1}}$ presented in the second case of Theorem 5.2.3, we will now present an example of a particular piecewise continuous function ψ (with symmetric discontinuities) for which we compute the winding number of $\psi\widetilde{\psi^{-1}}$ based on the winding number of a piecewise continuous function which depends on ψ, but having only discontinuities in $\mathbb{R}_-$.

Example 5.2.7. Consider the piecewise continuous function ψ (represented in Figure 5.9) given by

$$\psi(x) = \begin{cases} 1 + \dfrac{1-i}{x} & \text{if} \quad x \leq -1 \\ -x + i\,\dfrac{x+1}{2} & \text{if} \quad -1 < x \leq 1 \\ 1 + \dfrac{10-i}{x^3} & \text{if} \quad x > 1 \end{cases}. \tag{5.2.111}$$

It is clear that ψ is a continuous function at 0, and has symmetric discontinuities since ψ is discontinuous at -1 and 1. Additionally,

$$\left(\psi\widetilde{\psi^{-1}}\right)(x) = \begin{cases} \dfrac{x^3 + x^2(1-i)}{x^3 - 10 + i} & \text{if} \quad x < -1 \\ \dfrac{i}{i-1} & \text{if} \quad x = -1 \\ \dfrac{x(i-2)+i}{x(2-i)+i} & \text{if} \quad -1 < x < 1 \\ 1+i & \text{if} \quad x = 1 \\ \dfrac{x^3 + 10 - i}{x^3 + x^2(i-1)} & \text{if} \quad x > 1 \end{cases} \tag{5.2.112}$$

and therefore we may identify $\psi\widetilde{\psi^{-1}}$ with

$$\left(\psi\widetilde{\psi^{-1}}\right)(x) = \begin{cases} \dfrac{x^3+x^2(1-i)}{x^3-10+i} & \text{if} \quad x \leq -1 \\ \dfrac{x(i-2)+i}{x(2-i)+i} & \text{if} \quad -1 < x \leq 1 \\ \dfrac{x^3+10-i}{x^3+x^2(i-1)} & \text{if} \quad x > 1 \end{cases} . \tag{5.2.113}$$

From Figures 5.10, 5.11, 5.12, and 5.13, we may observe that

$$\operatorname{wind}\left(\psi\widetilde{\psi^{-1}}\right)^{\#} \neq \operatorname{wind}\left(\psi^{\#}\left(\widetilde{\psi^{-1}}\right)^{\#}\right), \tag{5.2.114}$$

i.e.,

$$\operatorname{wind}\left(\psi\widetilde{\psi^{-1}}\right)^{\#} \neq \operatorname{wind}\psi^{\#} + \operatorname{wind}\left(\widetilde{\psi^{-1}}\right)^{\#}. \tag{5.2.115}$$

Recalling that $\operatorname{wind}\left(\widetilde{\psi^{-1}}\right)^{\#} = \operatorname{wind}\psi^{\#}$, it follows that

$$\operatorname{wind}\left(\psi\widetilde{\psi^{-1}}\right)^{\#} \neq 2\operatorname{wind}\psi^{\#}, \tag{5.2.116}$$

which gives us

$$\operatorname{wind}\left(\psi\widetilde{\psi^{-1}}\right) \neq 2\operatorname{wind}\psi \tag{5.2.117}$$

(by using the definition of winding number of a piecewise continuous function). From the definition of ϱ_ψ (see (5.2.41)) we have

$$\varrho_\psi(x) = \begin{cases} \dfrac{i}{2}\dfrac{x^3+x^2(1-i)}{x^3-10+i} & \text{if} \quad x \leq -1 \\ \dfrac{i}{2}\dfrac{x(i-2)+i}{x(2-i)+i} & \text{if} \quad -1 < x \leq 0 \\ \dfrac{i}{2} & \text{if} \quad x > 0 \end{cases} . \tag{5.2.118}$$

According to Figure 5.14, we see that $\operatorname{wind}\varrho_\psi = 1$. Thus, from (5.2.72) it follows that $\operatorname{wind}\left(\psi\widetilde{\psi^{-1}}\right) = 2$ (which is also confirmed by Figure 5.12).

We close this chapter with a result that provides the invertibility of the Wiener-Hopf--Hankel operators $(W \pm H)_\phi$ in terms of the winding number of $\phi\widetilde{\phi^{-1}}$. This is in fact

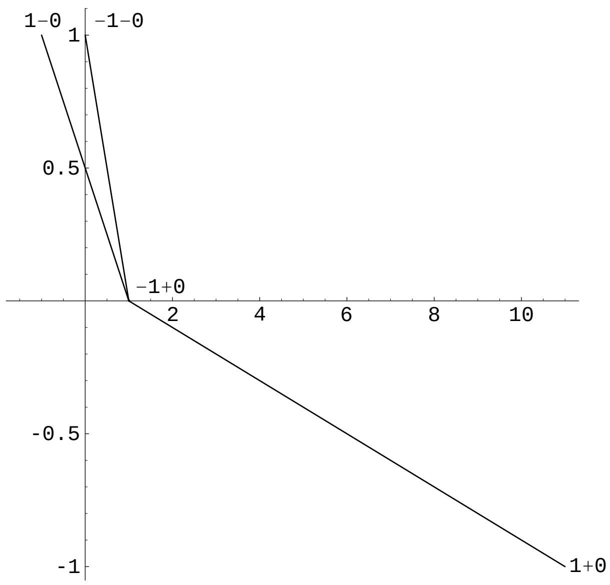


Figure 5.9: The range of $\psi(x)$ defined in (5.2.111) for x between -10000 and 10000.

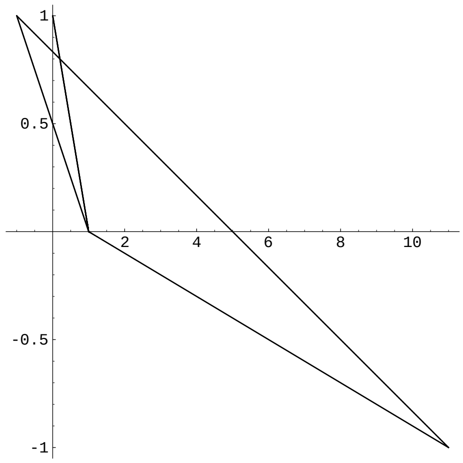


Figure 5.10: The graph of $\psi^{\#}$, for ψ defined in (5.2.111).

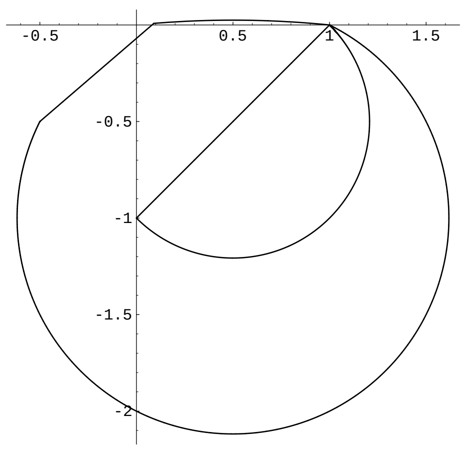


Figure 5.11: The graph of $\left(\widetilde{\psi^{-1}}\right)^{\#}$, for ψ defined in (5.2.111).

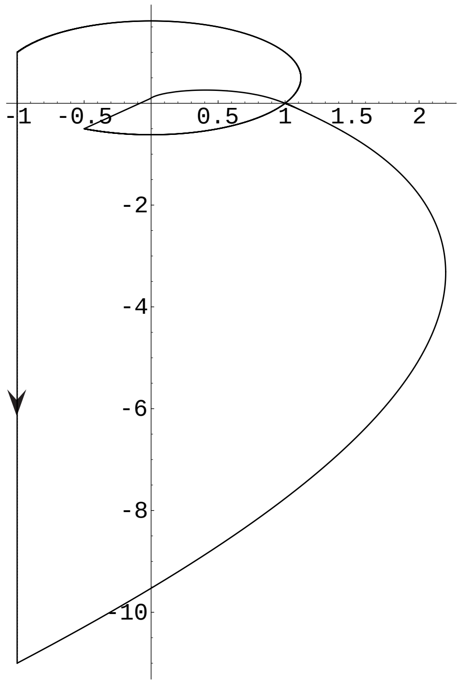


Figure 5.12: The oriented graph of $\left(\psi\widetilde{\psi^{-1}}\right)^{\#}$, for $\psi\widetilde{\psi^{-1}}$ defined in (5.2.113).

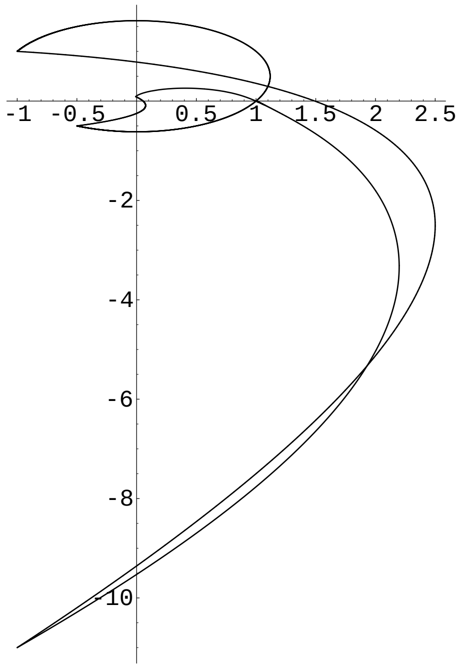


Figure 5.13: The graph of $\psi^{\#}\left(\widetilde{\psi^{-1}}\right)^{\#}$, for ψ defined in (5.2.111).

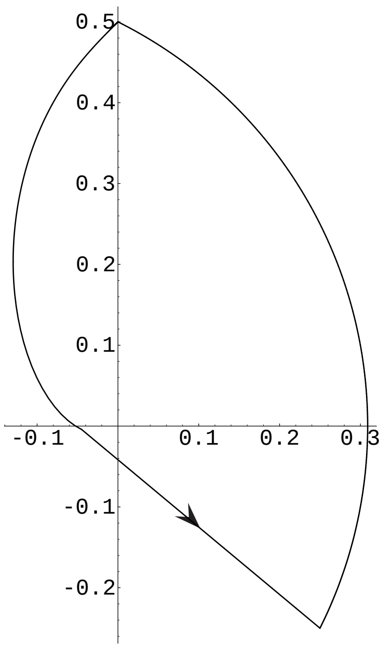


Figure 5.14: The oriented graph of $\varrho_{\psi}^{\#}$, for ϱ_{ψ} defined in (5.2.118).

an interesting result since, with the help of the Δ–relation after extension between the operators $(W+H)_\phi$ and $W_{\phi\widetilde{\phi^{-1}}}$, we can conclude that if

$$\mathrm{Ind}\,(W+H)_\phi + \mathrm{Ind}\,(W-H)_\phi = 0, \tag{5.2.119}$$

then $(W\pm H)_\phi$ are invertible operators (for piecewise almost periodic Fourier symbols).

Theorem 5.2.8. *If $\phi \in \mathcal{G}PAP$, $\kappa_l(\phi) + \kappa_r(\phi) = 0$,*

$$0 \notin \left[\frac{\mathbf{d}_l(\phi)}{\mathbf{d}_r(\phi)}, \frac{\mathbf{d}_r(\phi)}{\mathbf{d}_l(\phi)}\right] \cup \bigcup_{x\in\mathbb{R}} \left[(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)\right] \tag{5.2.120}$$

and $\mathrm{wind}\left(\phi\widetilde{\phi^{-1}}\right) = 0$*, then $(W\pm H)_\phi$ are invertible operators.*

Proof. Under the conditions of the statement, we have a scalar Wiener-Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ with zero Fredholm index (cf. Theorem 5.2.1). Thus, from the *Coburn-Simonenko Theorem* (see [24, 71]), we derive that $W_{\phi\widetilde{\phi^{-1}}}$ is invertible. Therefore, Corollary 1.3.8 and Corollary 1.3.10 ensure that $(W\pm H)_\phi$ are invertible operators. □

From the last result we conclude now that the operators $(W\pm H)_\phi$ presented in Example 5.2.6 are invertible Wiener-Hopf-Hankel operators (although W_ϕ is not an invertible operator). Moreover, we can also conclude that an analogue of the Coburn-Simonenko Theorem (cited above) does not hold for Wiener-Hopf-Hankel operators since in Example 5.2.6 we have $\mathrm{wind}\,\phi = 1$ and the Wiener-Hopf-Hankel operators presented there are invertible. In this sense, we may consider Theorem 5.2.8 a Coburn-Simonenko type theorem for Wiener-Hopf-Hankel operators with piecewise almost periodic Fourier symbols.

Chapter 6

Matrix Wiener-Hopf-Hankel Operators with Good Hausdorff Sets

The present chapter deals with a generalization of the invertibility and semi-Fredholm criteria presented in Section 3.2. In this case, the generalization is in terms of considering Wiener-Hopf-Hankel operators with matrix APW Fourier symbols and having a particular Hausdorff set bounded away from zero. The motivation for that is a result of R. G. Babadzhanyan and V. S. Rabinovich that settles the (one-sided and both-sided) invertibility and the semi-Fredholm property of Wiener-Hopf operators with matrix APW Fourier symbols having Hausdorff sets bounded away from zero.

6.1 Preliminaries

Let Φ be a $n \times n$ matrix function with elements belonging to $L^\infty(\mathbb{R})$, i.e., $\Phi \in [L^\infty(\mathbb{R})]^{n\times n}$. In this chapter, we will consider matrix Wiener-Hopf plus Hankel operators given by

$$(W+H)_\Phi := W_\Phi + H_\Phi \ : \ [L^2_+(\mathbb{R})]^n \to [L^2(\mathbb{R}_+)]^n \tag{6.1.1}$$

and matrix Wiener-Hopf minus Hankel operators given by

$$(W-H)_\Phi := W_\Phi - H_\Phi \ : \ [L^2_+(\mathbb{R})]^n \to [L^2(\mathbb{R}_+)]^n \ , \tag{6.1.2}$$

with W_Φ and H_Φ being matrix Wiener-Hopf and Hankel operators defined by

$$W_\Phi := r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F} : [L^2_+(\mathbb{R})]^n \to [L^2(\mathbb{R}_+)]^n , \tag{6.1.3}$$

$$H_\Phi := r_+\mathcal{F}^{-1}\Phi\cdot\mathcal{F}J : [L^2_+(\mathbb{R})]^n \to [L^2(\mathbb{R}_+)]^n , \tag{6.1.4}$$

respectively.

As about the notations, here and in what follows, for a given set A, we use A^n and $A^{n\times n}$ to denote the columns of length n and the $n\times n$ matrices with entries in A, respectively. Moreover, r_+ represents the operator of restriction from $[L^2_+(\mathbb{R})]^n$ into $[L^2(\mathbb{R}_+)]^n$, and J is the *reflection operator on* $\mathbb{R}$ defined by $J\Psi(x) = \widetilde{\Psi}(x) = \Psi(-x)$, for matrices Ψ.

To achieve the invertibility and semi-Fredholm criteria, we will use the same technique as the one used before in Chapters 4 and 5, i.e., we will use the Δ–relation after extension between Wiener-Hopf plus Hankel operators and Wiener-Hopf operators. Since the results for matrix Wiener-Hopf-Hankel operators (as well as their proofs) are similar to the results presented in Section 1.3.3, here we only present the results without giving their proofs.

Lemma 6.1.1. *Let $\Phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{n\times n}$. The matrix Wiener-Hopf plus Hankel operator*

$$(W+H)_\Phi : [L^2_+(\mathbb{R})]^n \to [L^2(\mathbb{R}_+)]^n \tag{6.1.5}$$

is Δ–related after extension with the Wiener-Hopf operator

$$W_{\Phi\widetilde{\Phi^{-1}}} : [L^2_+(\mathbb{R})]^n \to [L^2(\mathbb{R}_+)]^n \tag{6.1.6}$$

(with the same matrix size as the original one), being the Δ–relation after extension made the with the help of the auxiliary operator

$$\mathcal{T}_\Phi := \mathcal{F}^{-1}(\Phi\cdot - \Phi\cdot J)\mathcal{F}P_+ + P_- : [L^p(\mathbb{R})]^n \to [L^p(\mathbb{R})]^n . \tag{6.1.7}$$

Corollary 6.1.2. *Let $\Phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{n\times n}$. If the Wiener-Hopf operator $W_{\Phi\widetilde{\Phi^{-1}}}$ is invertible, left-invertible, right-invertible, Fredholm, n-normal, d-normal or normally solvable, then the Wiener-Hopf plus Hankel operator $(W+H)_\Phi$ has the same property as $W_{\Phi\widetilde{\Phi^{-1}}}$.*

Proposition 6.1.3. *Let $\Phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{n\times n}$. The operator $\mathcal{T}_\Phi$ is equivalent after extension to the Wiener-Hopf minus Hankel operator $(W-H)_\Phi$.*

Corollary 6.1.4. *Let $\Phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{n\times n}$. If the Wiener-Hopf operator $W_{\Phi\widetilde{\Phi^{-1}}}$ is invertible, left-invertible, right-invertible, Fredholm, n-normal, d-normal or normally solvable, then the Wiener-Hopf minus Hankel operator $(W-H)_\Phi$ has the same property as $W_{\Phi\widetilde{\Phi^{-1}}}$.*

6.2 Hausdorff sets

The *Hausdorff set* (or *numerical range*) of a complex matrix $\Theta \in \mathbb{C}^{n\times n}$ is defined as

$$\mathcal{H}(\Theta) := \{(\Theta\eta, \eta) : \eta \in \mathbb{C}^n, \|\eta\| = 1\}. \tag{6.2.8}$$

If $\Phi \in [APW]^{n\times n}$, then (due to the definition of APW) it holds that $\mathcal{H}(\Phi(x))$ is a well--defined subset of $\mathbb{C}^n$ for all $x \in \mathbb{R}$. In this way, the Hausdorff set of Φ is said to be *bounded away from zero* (or said to be a "good" Hausdorff set) if

$$\inf_{x\in\mathbb{R}} \operatorname{dist}\Big(\mathcal{H}\big(\Phi(x)\big),\ 0\Big) > 0 \tag{6.2.9}$$

or, equivalently, if there is an $\varepsilon > 0$ such that

$$\big|\big(\Phi(x)\eta, \eta\big)\big| \geq \varepsilon\|\eta\|^2 \quad \text{for all } x \in \mathbb{R} \text{ and all } \eta \in \mathbb{C}^n. \tag{6.2.10}$$

Consider $\Phi \in [APW]^{n\times n}$ and $\eta \in \mathbb{C}^n\backslash\{0\}$. If the Hausdorff set of Φ is bounded away from zero, then the function $(\Phi\eta, \eta)$ given by

$$(\Phi\eta, \eta)(x) := (\Phi(x)\eta, \eta)\ , \quad x \in \mathbb{R}, \tag{6.2.11}$$

is invertible in APW. Therefore, the mean motion of $(\Phi\eta, \eta)$, denoted by $\kappa\big((\Phi\eta, \eta)\big)$, is well-defined for all $\eta \in \mathbb{C}^n\backslash\{0\}$. In [1, 2], R. G. Babadzhanyan and V. S. Rabinovich proved that $\kappa\big((\Phi\eta, \eta)\big)$ is independent of $\eta \in \mathbb{C}^n\backslash\{0\}$, as we can see in the next result.

Theorem 6.2.1. ([1, 2]) *If $\Phi \in \mathcal{G}[APW]^{n\times n}$ and the Hausdorff set of Φ is bounded away from zero, then Φ has a right APW factorization and all right AP indices are equal to $\kappa((\Phi\,\eta, \eta))$ where η is an arbitrary vector in $\mathbb{C}^n\backslash\{0\}$.*

As a consequence of this last result, R. G. Babadzhanyan and V. S. Rabinovich presented in [1, 2] a criterion that settles the (one-sided and both-sided) invertibility and the semi-Fredholm property of Wiener-Hopf operators with matrix APW Fourier symbols having Hausdorff sets bounded away from zero, based on the value of the mean motion of a particular scalar APW function, $\kappa\big((\Phi\eta,\eta)\big)$. This result is in the origin of this chapter and states the following:

Theorem 6.2.2. ([1, 2]) *Let $\Phi \in \mathcal{G}[APW]^{n\times n}$ such that the Hausdorff set of Φ is bounded away from zero.*

(a) *If $\kappa((\Phi\,\eta,\eta)) = 0$, then W_Φ is invertible.*

(b) *If $\kappa((\Phi\,\eta,\eta)) > 0$, then W_Φ is properly n-normal and left-invertible.*

(c) *If $\kappa((\Phi\,\eta,\eta)) < 0$, then W_Φ is properly d-normal and right-invertible.*

In the next section, we will obtain a generalization of this result for Wiener-Hopf-Hankel operators with matrix APW Fourier symbols.

6.3 Invertibility and semi-Fredholm criteria

Theorem 6.3.1. *Let $\Phi \in \mathcal{G}[APW]^{n\times n}$ such that the Hausdorff set of $\Phi\widetilde{\Phi^{-1}}$ is bounded away from zero.*

(a) *If $\kappa\big((\Phi\widetilde{\Phi^{-1}}\,\eta,\eta)\big) = 0$, then $(W+H)_\Phi$ and $(W-H)_\Phi$ are invertible.*

(b) *If $\kappa\big((\Phi\widetilde{\Phi^{-1}}\,\eta,\eta)\big) > 0$, then $(W+H)_\Phi$ and $(W-H)_\Phi$ are left-invertible, and at least one of these operators is properly n-normal.*

(c) *If $\kappa\big((\Phi\widetilde{\Phi^{-1}}\,\eta,\eta)\big) < 0$, then $(W+H)_\Phi$ and $(W-H)_\Phi$ are right-invertible, and at least one of these operators is properly d-normal.*

Proof. The statement is a consequence of the Δ–relation after extension presented in Lemma 6.1.1, and of the corresponding result for Wiener-Hopf operators (cf. Theorem 6.2.2).

In fact, first, the hypothesis in (a), (b), and (c) give us the invertibility, left-invertibility, and right-invertibility of $W_{\Phi\widetilde{\Phi^{-1}}}$, respectively. Secondly, by using Corollary 6.1.2 and Corollary 6.1.4, we obtain that $(W+H)_\Phi$ and $(W-H)_\Phi$ are invertible, left-invertible, and right-invertible, under the conditions of case (a), (b) and (c), respectively. Finally, interpreting the Δ–relation after extension between the Wiener-Hopf plus Hankel operator $(W+H)_\Phi$ and the Wiener-Hopf operator $W_{\Phi\widetilde{\Phi^{-1}}}$ (cf. Lemma 6.1.1) as an equivalence after extension between $\mathrm{diag}[(W+H)_\Phi, \mathcal{T}_\Phi]$ and $W_{\Phi\widetilde{\Phi^{-1}}}$, and applying then the equivalence after extension between the operators $\mathcal{T}_\Phi$ and $(W-H)_\Phi$ (cf. Proposition 6.1.3), we reach to the final conclusion about the Wiener-Hopf-Hankel operators. □

Remark 6.3.2. We would like to point out that the condition of the Hausdorff set of $\Phi\widetilde{\Phi^{-1}}$ be bounded away from zero is obviously a fundamental condition in our main result, and it is clear that not every matrix function in $\mathcal{G}[APW]^{n\times n}$ has this property. For instance,

$$\Phi(x) = \begin{bmatrix} 2e^{e^{ix}} & e^{e^{ix}} \\ e^{-e^{ix}} & e^{-e^{ix}} \end{bmatrix}, \quad x \in \mathbb{R}, \tag{6.3.12}$$

is invertible in $[APW]^{n\times n}$ but produces a matrix function $\Phi\widetilde{\Phi^{-1}}$ which does not have a Hausdorff set bounded away from zero. Indeed, computing $\Phi\widetilde{\Phi^{-1}}$ we obtain

$$(\Phi\widetilde{\Phi^{-1}})(x) = \begin{bmatrix} e^{e^{ix}-e^{-ix}} & 0 \\ 0 & e^{e^{-ix}-e^{ix}} \end{bmatrix}, \quad x \in \mathbb{R}, \tag{6.3.13}$$

and considering $\eta = (\eta_1, \eta_2)^\top \in \mathbb{C}^2$, such that $\|\eta\| = 1$, it follows

$$\begin{aligned} \left(\left(\Phi\widetilde{\Phi^{-1}}\right)(x)\,\eta, \eta\right) &= \left(\begin{bmatrix} \eta_1\, e^{e^{ix}-e^{-ix}} \\ \eta_2\, e^{e^{-ix}-e^{ix}} \end{bmatrix}, \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \right) \\ &= |\eta_1|^2 e^{e^{ix}-e^{-ix}} + |\eta_2|^2 e^{e^{-ix}-e^{ix}}. \end{aligned} \tag{6.3.14}$$

Thus,

$$\mathcal{H}\left(\left(\Phi\widetilde{\Phi^{-1}}\right)(x)\right) = \left\{ |\eta_1|^2 e^{e^{ix}-e^{-ix}} + |\eta_2|^2 e^{e^{-ix}-e^{ix}} : \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \in \mathbb{C}^2, \|\eta\| = 1 \right\}, \tag{6.3.15}$$

and

$$\operatorname{dist}\left(\mathcal{H}\left(\left(\Phi\widetilde{\Phi^{-1}}\right)(x)\right),0\right) = \left||\eta_1|^2 e^{e^{ix}-e^{-ix}} + |\eta_2|^2 e^{e^{-ix}-e^{ix}}\right|, \tag{6.3.16}$$

with $\eta = (\eta_1, \eta_2)^\top \in \mathbb{C}^2$ such that $\|\eta\| = 1$. Since

$$\operatorname{dist}\left(\mathcal{H}\left(\left(\Phi\widetilde{\Phi^{-1}}\right)(x)\right),0\right) = 0 \Leftrightarrow \eta_1 = \pm\frac{1}{\sqrt{2}} \vee \eta_2 = \pm\frac{1}{\sqrt{2}}\,, \tag{6.3.17}$$

we conclude that the Hausdorff set of $\Phi\widetilde{\Phi^{-1}}$ is not bounded away from zero.

Considering $\Phi \in \mathcal{G}[APW]^{n\times n}$ such that the Hausdorff set of $\Phi\widetilde{\Phi^{-1}}$ is bounded away from zero, from Theorem 6.2.1, we have that $\Phi\widetilde{\Phi^{-1}}$ admits a right APW factorization. Combining this fact with analogue versions for matrix Wiener-Hopf-Hankel operators of some of the results presented in Chapters 1 and 3, we will now see how to obtain the "properly" limit case for both Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators in Theorem 6.3.1.

Theorem 6.3.3. *Let $\Phi \in \mathcal{G}[APW]^{n\times n}$ such that the Hausdorff set of $\Phi\widetilde{\Phi^{-1}}$ is bounded away from zero. Then $\Phi\widetilde{\Phi^{-1}}$ admits a right APW factorization of the form*

$$\left(\Phi\widetilde{\Phi^{-1}}\right)(x) = \Psi_-(x)\ \operatorname{diag}\left[e_{\kappa\left((\Phi\widetilde{\Phi^{-1}}\,\eta,\eta)\right)}\right]\ \Psi_+(x), \tag{6.3.18}$$

where $\Psi_- \in \mathcal{G}[APW^-]^{n\times n}$, $\Psi_+ \in \mathcal{G}[APW^+]^{n\times n}$, and η is an arbitrary vector in $\mathbb{C}^n\backslash\{0\}$. If $\Psi_+ = \widetilde{\Psi_-^{-1}}$ and

(a) $\kappa\left((\Phi\widetilde{\Phi^{-1}}\,\eta,\eta)\right) = 0$, *then $(W+H)_\Phi$ and $(W-H)_\Phi$ are invertible.*

(b) $\kappa\left((\Phi\widetilde{\Phi^{-1}}\,\eta,\eta)\right) > 0$, *then $(W+H)_\Phi$ and $(W-H)_\Phi$ are properly n-normal and left--invertible.*

(c) $\kappa\left((\Phi\widetilde{\Phi^{-1}}\,\eta,\eta)\right) < 0$, *then $(W+H)_\Phi$ and $(W-H)_\Phi$ are properly d-normal and right--invertible.*

Proof. Since $\Phi\widetilde{\Phi^{-1}} \in \mathcal{G}[APW]^{n\times n}$ is such that the Hausdorff set of $\Phi\widetilde{\Phi^{-1}}$ is bounded away from zero, according to Theorem 6.2.1 it follows that $\Phi\widetilde{\Phi^{-1}}$ admits a right APW

factorization of the form

$$\left(\Phi\widetilde{\Phi^{-1}}\right)(x) = \Psi_{-}(x)\ \operatorname{diag}\left[e_{\kappa\left((\Phi\widetilde{\Phi^{-1}}\,\eta,\eta)\right)}\right]\ \Psi_{+}(x), \tag{6.3.19}$$

where $\Psi_{-} \in \mathcal{G}[APW^{-}]^{n\times n}$, $\Psi_{+} \in \mathcal{G}[APW^{+}]^{n\times n}$, and η is an arbitrary vector in $\mathbb{C}^n\backslash\{0\}$. Considering $\Psi_{+} = \widetilde{\Psi_{-}^{-1}}$, we obtain

$$\left(\Phi\widetilde{\Phi^{-1}}\right)(x) = \Psi_{-}(x)\ \operatorname{diag}\left[e_{\kappa\left((\Phi\widetilde{\Phi^{-1}}\,\eta,\eta)\right)}\right]\ \widetilde{\Psi_{-}^{-1}}(x), \tag{6.3.20}$$

which means that $\Phi\widetilde{\Phi^{-1}}$ admits an APW antisymmetric factorization (see (3.1.11)). Using the matrix version of Proposition 3.1.5, we conclude that Ψ admits an APW asymmetric factorization:

$$\Phi(x) = \Psi_{-}(x)\ \operatorname{diag}\left[e_{\frac{1}{2}\kappa\left((\Phi\widetilde{\Phi^{-1}}\,\eta,\eta)\right)}\right]\ \Psi_{e}(x), \tag{6.3.21}$$

with

$$\Psi_e = e_{-\frac{1}{2}\kappa\left((\Phi\widetilde{\Phi^{-1}}\,\eta,\eta)\right)}\,\Psi_{-}^{-1}\Phi. \tag{6.3.22}$$

Applying now the analogue of Propositions 1.3.4 and 1.3.5 and Remark 1.3.6 for matrix Wiener-Hopf-Hankel operators, it holds that:

(a) $(W+H)_{\Phi}$ is equivalent to $(W+H)_{\operatorname{diag}\left[e_{\frac{1}{2}\kappa\left((\Phi\widetilde{\Phi^{-1}}\,\eta,\eta)\right)}\right]}$,

(b) and $(W-H)_{\Phi}$ is equivalent to $(W-H)_{\operatorname{diag}\left[e_{\frac{1}{2}\kappa\left((\Phi\widetilde{\Phi^{-1}}\,\eta,\eta)\right)}\right]}$.

The assertion follows now from Theorem 3.2.1 and Remark 3.3.4. □

Remark 6.3.4. We would like to point out that Theorems 6.3.1 and 6.3.3 are a generalization of the invertibility and semi-Fredholm criteria presented in Chapter 3 for Wiener--Hopf-Hankel operators with scalar APW Fourier symbols. Moreover, we would like also to stress that Theorem 6.3.3, being the "properly" limit case of Theorem 6.3.1, is achieved with an extra condition about the APW antisymmetric factorization of $\Phi\widetilde{\Phi^{-1}}$, $\Psi_{+} = \widetilde{\Psi_{-}^{-1}}$. Like in the scalar case, the APW antisymmetric factorization of $\Phi\widetilde{\Phi^{-1}}$ is related with the APW asymmetric factorization of Φ. However, since an analogue of Theorem 3.1.3

does not hold for $[APW]^{n\times n}$ functions and consequently we cannot guarantee that every $\Phi \in \mathcal{G}[APW]^{n\times n}$ admits an APW asymmetric factorization, we cannot "remove" the condition about the APW antisymmetric factorization of $\Phi\widetilde{\Phi^{-1}}$ (or equivalently, the condition about the APW asymmetric factorization of Φ) presented in Theorem 6.3.3 like in the case of Theorem 3.2.1 (where a condition about the APW asymmetric factorization of the Fourier symbol of the operators does not appear in the statement of the theorem).

6.4 An example

To illustrate Theorem 6.3.1, we will present in this last section a concrete case of invertible matrix Wiener-Hopf-Hankel operators $(W \pm H)_{\Phi_p}$ with an $[APW]^{2\times 2}$ Fourier symbol Φ_p.

Example 6.4.1. Let us consider the particular matrix-valued function

$$\Phi_p(x) = \begin{bmatrix} 2e^{e^{ix}} & e^{e^{ix}-1} \\ e^{e^{-i3x}+1} & e^{e^{-i3x}} \end{bmatrix}, \quad x \in \mathbb{R}. \tag{6.4.23}$$

From the reformulation of Bohr's theorem to the class $\mathcal{G}APW$ mentioned before (cf. (3.1.19)), it follows that $\Phi_p \in [APW]^{2\times 2}$. Moreover, Φ_p is invertible and we have

$$\Phi_p^{-1}(x) = \begin{bmatrix} e^{-e^{ix}} & -e^{-e^{-i3x}-1} \\ -e^{-e^{ix}+1} & 2e^{-e^{-i3x}} \end{bmatrix}, \quad x \in \mathbb{R}. \tag{6.4.24}$$

It can be easily seen that $\Phi_p^{-1} \in [APW]^{2\times 2}$, which yields that $\Phi_p \in \mathcal{G}[APW]^{2\times 2}$. As about the form of $\Phi_p\widetilde{\Phi_p^{-1}}$, we have in this case

$$(\Phi_p\widetilde{\Phi_p^{-1}})(x) = \begin{bmatrix} e^{e^{ix}-e^{-ix}} & 0 \\ 0 & e^{e^{-i3x}-e^{i3x}} \end{bmatrix}, \quad x \in \mathbb{R}. \tag{6.4.25}$$

We will now show that the Hausdorff set of $\Phi_p\widetilde{\Phi_p^{-1}}$ is bounded away from zero. Considering

$\eta = (\eta_1, \eta_2)^\top \in \mathbb{C}^2$, such that $\|\eta\| = 1$, it follows that

$$\left(\left(\Phi_p \widetilde{\Phi_p^{-1}} \right)(x) \, \eta, \eta \right) = \left(\begin{bmatrix} \eta_1 \, e^{\, e^{ix} - e^{-ix}} \\ \eta_2 \, e^{\, e^{-i3x} - e^{i3x}} \end{bmatrix}, \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \right)$$
$$= |\eta_1|^2 e^{\, e^{ix} - e^{-ix}} + |\eta_2|^2 e^{\, e^{-i3x} - e^{i3x}}, \quad x \in \mathbb{R}. \tag{6.4.26}$$

Therefore,

$$\mathcal{H}\left(\left(\Phi_p \widetilde{\Phi_p^{-1}} \right)(x) \right)$$
$$= \left\{ |\eta_1|^2 e^{\, e^{ix} - e^{-ix}} + |\eta_2|^2 e^{\, e^{-i3x} - e^{i3x}} : \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \in \mathbb{C}^2, \|\eta\| = 1 \right\}, x \in \mathbb{R}. \tag{6.4.27}$$

Then, we have that

$$\operatorname{dist}\left(\mathcal{H}\left(\left(\Phi_p \widetilde{\Phi_p^{-1}} \right)(x) \right), 0 \right) = \left| |\eta_1|^2 e^{\, e^{ix} - e^{-ix}} + |\eta_2|^2 e^{\, e^{-i3x} - e^{i3x}} \right|, \tag{6.4.28}$$

with $\eta = (\eta_1, \eta_2)^\top \in \mathbb{C}^2$ such that $\|\eta\| = 1$. Let us first analyze the particular case when $|\eta_1| = 1$ (and $\eta_2 = 0$). In such a case, it is clear that $\operatorname{dist}\left(\mathcal{H}\left(\left(\Phi_p \widetilde{\Phi_p^{-1}} \right)(x) \right), 0 \right) = 1$, for all $x \in \mathbb{R}$. Assume now that $0 < |\eta_1| < 1$. From (6.4.28), we have

$$\begin{aligned} \operatorname{dist}\left(\mathcal{H}\left(\left(\Phi_p \widetilde{\Phi_p^{-1}} \right)(x) \right), 0 \right) &= \left| |\eta_1|^2 e^{\, i2\sin(x)} + |\eta_2|^2 e^{\, -i2\sin(3x)} \right| \\ &= \left| |\eta_1|^2 e^{\, -i2\sin(3x)} \left(e^{\, i2(\sin(x) + \sin(3x))} + c \right) \right| \\ &= |\eta_1|^2 \left| e^{\, i2(\sin(x) + \sin(3x))} + c \right|, \end{aligned} \tag{6.4.29}$$

where

$$c = \frac{1 - |\eta_1|^2}{|\eta_1|^2} > 0. \tag{6.4.30}$$

We will now verify that

$$\left| e^{\, i2(\sin(x) + \sin(3x))} + c \right| \neq 0, \tag{6.4.31}$$

for all $x \in \mathbb{R}$ and $c > 0$. Considering the notation

$$\theta = 2(\sin(x) + \sin(3x)), \tag{6.4.32}$$

we have

$$\left| e^{\,i2(\sin(x)+\sin(3x))} + c \right|^2 = (\cos(\theta)+c)^2 + \sin^2(\theta) = 1 + 2c\cos(\theta) + c^2 \qquad (6.4.33)$$

and notice that $c^2 + 2c\cos(\theta) + 1 = 0$ if and only if $c = -\cos(\theta) \pm i|\sin(\theta)|$. Therefore, on one hand, if $\theta = 0$ we obtain $c = -1$, and on the other hand, if $\theta \neq 0$ then c is a complex (non-real) number. Both cases are impossible due to (6.4.30). Altogether, we obtain

$$\inf_{x\in\mathbb{R}} \operatorname{dist}\left(\mathcal{H}\left(\left(\Phi_p \widetilde{\Phi_p^{-1}}\right)(x) \right), 0 \right) > 0, \qquad (6.4.34)$$

i.e., the Hausdorff set of $\Phi_p\widetilde{\Phi_p^{-1}}$ is bounded away from zero (or “good”).

Now for computing the corresponding mean motion, we start by considering $\eta = (1,0)^\top$. From (6.4.26), it follows that

$$\left(\left(\Phi_p \widetilde{\Phi_p^{-1}}\right)(x)\, \eta, \eta \right) = e^{\,e^{ix} - e^{-ix}}, \qquad x \in \mathbb{R}\,. \qquad (6.4.35)$$

Since $e^{ix} - e^{-ix} \in APW$, from the reformulation of Bohr's theorem to the class $\mathcal{G}APW$, it follows that

$$\kappa\left(\left(\left(\Phi_p \widetilde{\Phi_p^{-1}}\right) \eta, \eta \right) \right) = 0 \qquad (6.4.36)$$

for $\eta = (1,0)^\top$. Due to (6.4.34) and (6.4.36), and according to Theorem 6.2.1, we conclude that $\kappa\big(((\Phi_p\widetilde{\Phi_p^{-1}})\eta,\eta)\big) = 0$ for all $\eta \in \mathbb{C}^n \backslash \{0\}$. Finally, applying Theorem 6.3.1, we derive that $(W \pm H)_{\Phi_p}$ are in fact invertible operators.

Conclusion

Nowadays, the theory of Wiener-Hopf-Hankel operators is well developed for some classes of Fourier symbols. This is the case of continuous and piecewise continuous symbols, since they are of great importance for the applications. However, this is not the case for almost periodic and semi-almost periodic Fourier symbols which are also important in the view of applications. Due to this, in this thesis we studied the regularity properties of Wiener-Hopf-Hankel operators with Fourier symbols belonging to the algebras of almost periodic, semi-almost periodic and piecewise almost periodic functions, and acting between L^p Lebesgue spaces.

After have defined, in Chapter 1, the Wiener-Hopf-Hankel operators under study, and presented several relations between some classes of convolution type operators that had a fundamental role in the achievement of the invertibility and Fredholm criteria presented in Chapters 3, 4, 5 and 6, we devoted the second chapter to the algebras of almost periodic, semi-almost periodic and piecewise almost periodic functions. In Chapter 3 we proposed a factorization theory for the Wiener-Hopf-Hankel operators with almost periodic Fourier symbols acting in L^2 Lebesgue spaces. We introduced a factorization concept for almost periodic Fourier symbols such that the properties of the factors allow corresponding operator factorizations. A criterion for the semi-Fredholm property and for the one-sided and both-sided invertibility was obtained upon certain indices of the factorizations. Under those conditions, the one-sided and two-sided inverses of the operators were also obtained. Moreover, the introduced factorizations also allowed the exposition of dependencies between the invertibility of Wiener-Hopf and Wiener-Hopf-Hankel operators with the same

Fourier symbol.

Based on the delta relation after extension, in Chapter 4, we established a Sarason's type theorem for Wiener-Hopf-Hankel operators with semi-almost periodic Fourier symbols and acting between L^2 Lebesgue spaces. This means a characterization of the Fredholm property, and one-sided invertibility of these operators, based on the mean motions and geometric mean values of the almost periodic representatives of the Fourier symbols at minus and plus infinity. A formula for the Fredholm index was derived, and conditions for the invertibility of the operators in study were obtained. Considering then Wiener-Hopf-Hankel operators with semi-almost periodic Fourier symbols, acting between L^p Lebesgue spaces, we derived a generalization of the Sarason's type theorem, the so-called Duduchava-Saginashvili's type theorem. Additionally, a formula for the Fredholm index was provided by introducing a corresponding winding number of some new elements.

Chapter 5 was dedicated to Wiener-Hopf-Hankel operators with piecewise almost periodic Fourier symbols, acting between L^2 Lebesgue spaces. For these operators, a criterion for the Fredholm property and for the one-sided invertibility was also obtained upon the use of the delta relation after extension. Due to the nature of piecewise almost periodic functions, this criterion was based on the mean motions and geometric mean values of the almost periodic representatives of the Fourier symbols as well as on the discontinuities of certain auxiliary functions. A formula for the sum of the Fredholm indices of these Wiener-Hopf plus and minus Hankel operators was also derived, and interpreted upon different cases of symmetries of the discontinuities of the Fourier symbols.

Finally, in Chapter 6, we considered a generalization of Wiener-Hopf-Hankel operators with APW Fourier symbols acting in L^2 Lebesgue spaces presented in Chapter 3, i.e., we considered Wiener-Hopf-Hankel operators with matrix APW Fourier symbols. For these operators, we achieved an invertibility and semi-Fredholm criteria based on the assumption that a particular Hausdorff set is bounded away from zero.

As about open questions that remain to be answered, we would like to stress the following two. In the first chapter, we derive necessary conditions for the semi-Fredholm property of the Wiener-Hopf-Hankel operators. We concluded that if the Fourier symbol of the Wiener-Hopf-Hankel operator is not invertible in $\mathcal{M}^p(\mathbb{R})$, then the Wiener-Hopf--Hankel operator is not a semi-Fredholm operator. In the case of Wiener-Hopf operators, there is a stronger result that asserts that if the Fourier symbol of the operator is not invertible in $\mathcal{M}^p(\mathbb{R})$, then the Wiener-Hopf operator is not normally solvable. Although our intuition goes into the same direction, until now we were not able to show that if the Fourier symbol of the Wiener-Hopf-Hankel operator is not invertible in $\mathcal{M}^p(\mathbb{R})$, then the Wiener-Hopf-Hankel operator is not normally solvable.

The other question concerns to the Fredholm index formula for Fredholm Wiener-Hopf--Hankel operators with piecewise almost periodic symbols and acting between L^2 spaces. Recall that, in Chapter 5, we provided a formula for the sum of the Fredholm indices of the Wiener-Hopf plus and minus Hankel operators, this was obtained due to the Δ–relation after extension between the Wiener-Hopf plus Hankel and the Wiener-Hopf operators. There we have also shown that Wiener-Hopf plus and minus Hankel operators may have different Fredholm indices. Thus, the question of finding formulae for the Fredholm index of Wiener-Hopf plus Hankel operators and for the Fredholm index of Wiener-Hopf minus Hankel operators still remains to be solved.

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